Approximating Weakly Preferred Semantics in Abstract Argumentation through Vacuous Reduct Semantics - Proofs of Technical Results

Lydia Blümel , Matthias Thimm
Artificial Intelligence Group, FernUniversität in Hagen
lydia.bluemel@fernuni-hagen.de, matthias.thimm@fernuni-hagen.de

A Omitted Proofs of Section 3

Proposition 15. *For any AF $F = (A, R)$ it holds that*

$$vac_{ad}(cf) = \{E \in pr(F) | E \cup E^+ \cup \{a \in A | (a, a) \in R\} = A\}$$

Proof. ($\supseteq$) Since preferred extensions are admissible $E \in pr(F)$ covers the base condition. If $E \cup E^+ \cup \{a \in A|(a, a) \in R\} = A$ then $A \setminus \{a \in A|(a, a) \in R\} \subseteq E \cup E^+$. Therefore F^E contains only self-attackers, so $cf(F^E) = \{\emptyset\}$.

($\subseteq$) Suppose $E \in vac_{ad}(cf)(F)$. $.cf(F^E) = \{\emptyset\}$ implies $ad(F^E) = \{\emptyset\}$ so $vac_{ad}(cf)(F) \subseteq vac_{ad}(ad)(F) = pr(F)$. It also implies all arguments in F^E are self-attackers, which means all none-self-attackers in F are either in or attacked by E. □

Proposition 16. $vac_{cf}(cf) = stb_{\delta_{self}}$

Proof. Cogent stable extensions are conflictfree due to the generalized definition of refute-based stable semantics $stb_\delta(F) = \{E\ \delta - conflictfree | E \cup \delta(E) = A\}$ from (Blümel and Ulbricht 2022). Since $\delta_{self}(E) = E^+ \cup \{a \in A|(a, a) \in R\}$ the reduct of any such E contains only self-attackers i. e. no conflictfree sets besides $\emptyset$. For the other direction the vacuity condition guarantees $E \cup \delta_{self}(E) = A$ and the base condition that E is δ_{self}-conflictfree for any $E \in vac_{cf}(cf)$. □

B Omitted Proofs of Section 4

Proposition 17. *For any σ, τ argumentation semantics it holds that*

$$vac_\sigma(\tau) = vac_{vac_\sigma(\tau)}(\tau)$$

Proof. The direction $\supseteq$ is trivial. Suppose $E \in vac_\sigma(\tau)(F)$ for some AF $F = (A, R)$. Then $\tau(F^E) \subseteq \{\emptyset\}$. So E satisfies both the base and vacuity condition of members of $vac_{vac_\sigma(\tau)}(\tau)(F)$. □

Proposition 20. *Let $F = (A, R)$ be an AF. For any $n \in \mathbb{N}, n \geq 1$ it holds that:*

$$vac^n_{cf}(ad)(F) = vac^n_{cf}(pr)(F)$$

In particular the undisputed semantics is the $vac^1_{cf}(pr)$-semantics and the strongly undisputed semantics the $vac^2_{cf}(pr)$-semantics.

Proof. By Def. 18 we have $vac^n_{cf}(ad) = vac^n_{cf}(pr)$. For $n = 1$ Prop. 10 ensures $vac^1_{cf}(pr) = vac_cf(pr) = vac_{cf}(ad) = vac^1_{cf}(ad)$. With this as the base case induction over n yields $vac^n_{cf}(ad)(F) = \{E \in cf(F) \mid vac^{n-1}_{cf}(ad)(F^E) \subseteq \{\emptyset\}\} = \{E \in cf(F) \mid vac^{n-1}_{cf}(ad)(F^E) \subseteq \{\emptyset\}\} = vac^n_{cf}(ad)$. □

C Omitted Proofs of Section 5

Proposition 28. *Let $F = (A, R)$ an AF with $|A| = n$. Then for all $m > n$ $vac^m_{cf}(ad)(F) = {}^\infty cf(F)$.*

Proof. Proof by induction over size of $|A| = n$.

(Base case) For $F = (\emptyset, \emptyset)$ $vac^m_{cf}(ad)(F) = \{\emptyset\} = {}^\infty cf(F)$ for all $n \in \mathbb{N}$.

(Induction step) Suppose there existed an $E \in vac^m_{cf}(ad)(F) \setminus {}^\infty cf(F)$. Then there exists a nonempty $D \in {}^\infty cf(F^E) \setminus vac^{m-1}_{cf}(ad)(F^E)$. In case of $E \neq \emptyset$ this contradicts the induction hypothesis, since F^E has at most $n - 1$ arguments. The same is true for $E \in {}^\infty cf(F) \setminus vac^m_{cf}(ad)(F)$. For $E = \emptyset$ the assumption $E \in vac^m_{cf}(ad)(F) \setminus {}^\infty cf(F)$ implies the existence of a $D_1 \in {}^\infty cf(F) \setminus vac^{m-1}_{cf}(ad)(F)$, $D \neq \emptyset$. Which implies there exists a $D_2 \in vac^{m-2}_{cf}(ad)(F^{D_1}) \setminus {}^\infty cf(F^{D_1})$ and therefore a $D_3 \in {}^\infty cf(F^{D_1 \cup D_2}) \setminus vac^{m-3}_{cf}(ad)(F^{D_1 \cup D_2})$ must exist etc. up to the existence of a $D_{m-1} \in pr(F^{\bigcup D_i}) \setminus {}^\infty cf(F^{\bigcup D_i})$ (or the other way around, depending on m even or odd). Since all $D_i \neq \emptyset$ the $m-1$-th reduct $F^{\bigcup D_i}$ has at most $n - m + 1 \leq 0$ arguments, i. e. it is empty and therefore $pr(F^{\bigcup D_i}) = {}^\infty cf(F^{\bigcup D_i})$. Contradiction.

To be precise, for $E = \emptyset$ we use the following lemma:

Lemma 35. *Let $F = (A, R)$ be an AF such that for a fixed $m \in \mathbb{N}$: $vac^m_{cf}(ad)(F) \neq {}^\infty cf(F)$. Then for any $l < m$ there exists a sequence of argument sets $(E_0, ..., E_l)$ with $E_i \neq \emptyset$ an argument set of $F^{E_0 \cup ... \cup E_{i-1}}$ for all $1 \leq i \leq l$ ($E_0 \subseteq A$, E_0 may be empty) such that either $E_l \in vac^{m-l}_{cf}(ad)(F^{E_0 \cup ... \cup E_{l-1}}) \setminus {}^\infty cf(F^{E_0 \cup ... \cup E_{l-1}})$ or $E_l \in {}^\infty cf(F^{E_0 \cup ... \cup E_{l-1}}) \setminus vac^{m-l}_{cf}(ad)(F^{E_0 \cup ... \cup E_{l-1}})$ (depending on E_0 and whether l is even or odd).*

Proof. By induction over l. (Base case) $l = 0$ trivial, so assume $m > 1$ and $l = 1$. By assumption of the lemma there exists an $E_0 \subseteq A$ such that either $E_0 \in vac_{cf}^{m}(ad)(F) \setminus {}^{\infty}cf(F)$ or $E_0 \in {}^{\infty}cf(F) \setminus vac_{cf}^{m}(ad)(F)$. Since E_0 has to be conflictfree to be in either one of the two extension sets, it follows the vacuity condition is not satisfied for one of the two semantics, so there exists an $E_1 \neq \emptyset$ such that resp. either $E_1 \in {}^{\infty}cf(F^{E_0}) \setminus vac_{cf}^{m-1}(ad)(F^{E_0})$ or $E_1 \in vac_{cf}^{m-1}(ad)(F^{E_0}) \setminus {}^{\infty}cf(F^{E_0})$.

(Induction step) By the induction hypothesis there exists a sequence $(E_0, ..., E_{l-1})$ such that either $E_{l-1} \in vac_{cf}^{m-(l-1)}(ad)(F^{E_0\cup...\cup E_{l-2}}) \setminus {}^{\infty}cf(F^{E_0\cup...\cup E_{l-2}})$ or $E_{l-1} \in {}^{\infty}cf(F^{E_0\cup...\cup E_{l-2}}) \setminus vac_{cf}^{m-(l-1)}(ad)(F^{E_0\cup...\cup E_{l-2}})$. In this case E_{l-1} is conflictfree, so some $E_l \neq \emptyset$ must exist such that either $E_l \in vac_{cf}^{m-l}(ad)(F^{E_0\cup...\cup E_{l-1}}) \setminus {}^{\infty}cf(F^{E_0\cup...\cup E_{l-1}})$ or $E_l \in {}^{\infty}cf(F^{E_0\cup...\cup E_{l-1}}) \setminus vac_{cf}^{m-l}(ad)(F^{E_0\cup...\cup E_{l-1}})$ resp.. □

The proposition assumes $n < m$ so by the lemma there exists a sequence $(\emptyset, E_1, ..., E_n)$ with $E_i \neq \emptyset$ for all $i > 0$ such that either $E_n \in vac_{cf}^{m-n}(ad)(F^{E_0\cup...\cup E_{n-1}}) \setminus {}^{\infty}cf(F^{E_0\cup...\cup E_{n-1}})$ or $E_n \in {}^{\infty}cf(F^{E_0\cup...\cup E_{n-1}}) \setminus vac_{cf}^{m-n}(ad)(F^{E_0\cup...\cup E_{n-1}})$. But F has at most n arguments, so $F^{E_0\cup...\cup E_n}$ is empty and E_n thus either a member of both semantics or of neither of them. Contradiction. □

Theorem 30. *There exists a unique argumentation semantics τ_{cf}^{fp} satisfying $\tau_{cf}^{fp} = vac_{cf}(\tau_{cf}^{fp})$ and that is $\tau_{cf}^{fp} = {}^{\infty}cf$.*

Proof. (Part One) ${}^{\infty}cf = vac_{cf}({}^{\infty}cf)$. This follows from the definition of ${}^{\infty}cf$. By Def. 27 the semantics $vac_{cf}({}^{\infty}cf)$ is defined as $vac_{cf}({}^{\infty}cf) = \{E \in cf(F) \mid {}^{\infty}cf(F^E) \subseteq \{\emptyset\}\}$. For $E \neq \emptyset$ this is exactly the condition from Def. 27. For $E = \emptyset$ it holds that $\emptyset \in cf(F)$ and $F^{\emptyset} = F$. So $\emptyset \in {}^{\infty}cf(F)$ iff ${}^{\infty}cf(F) = \{\emptyset\}$ i.e. iff $\forall E \neq \emptyset,\ E \in cf(F) : E \notin {}^{\infty}cf(F)$ which is equivalent to the first part of Def. 27 $\forall E \neq \emptyset,\ E \in cf(F) : {}^{\infty}cf(F^E) \notin \{\emptyset, \{\emptyset\}\}$.

(Part Two) Proof of uniqueness. For any semantics τ if $\tau = vac_{cf}(\tau)$ then for any $F = (A, R)$ it holds that $\tau(F) = {}^{\infty}cf(F)$. Proof by induction over the number of arguments $|A| = n \in \mathbb{N}$.

(Base Case) Let $n = 0$ and thus $F = (\emptyset, \emptyset)$. Then $cf(F) = \emptyset$. Because $\tau = vac_{cf}(\tau)$ we thus know $\tau(F)$ is either the empty set or $\{\emptyset\}$. If $\tau = \{\emptyset\}$ then $\tau = {}^{\infty}cf$ and since $\tau(F^{\emptyset}) = \tau(F) = \{\emptyset\}$ we know $\emptyset \in vac_{cf}(\tau)(F)$ holds. so $\tau(F) = vac_{cf}(\tau)(F)$. Suppose $\tau = \emptyset$ on the other hand, then for the empty set we get $\tau(F^{\emptyset}) = \tau(F) = \emptyset \subseteq \{\emptyset\}$ so $\emptyset \in cf^{\tau}(F)$ and therefore $\tau(F) \neq cf^{\tau}(F)$. Contradiction with the assumption that $\tau = vac_{cf}(\tau)$. For $F = (\emptyset, \emptyset)$ we thus get the unique $\tau(F) = {}^{\infty}cf(F) = \{\emptyset\}$.

(Induction Step) Let $F = (A, R)$ with $|A| = n$. By the induction hypothesis for all $F' = (A', R')$ with $|A| < n$ it holds that $\tau(F') = {}^{\infty}cf(F')$ for any τ satisfying $\tau = vac_{cf}(\tau)$. We distinguish three cases. First, if $cf(F) = \{\emptyset\}$, then as explained in the base case $vac_{cf}(\tau)(F) \subseteq \{\emptyset\}$ so $E \in vac_{cf}(\tau)(F)$ so $\tau(F) = \{\emptyset\} = {}^{\infty}cf(F)$ is unique. Second, suppose some $E \in cf(F)$, $E \neq \emptyset$ exists. Then F^E has less arguments than F and thus satisfies the induction hypothesis, so $\tau(F^E) \subseteq \{\emptyset\}$ iff ${}^{\infty}cf(F^E) \subseteq \{\emptyset\}$. This implies $E \in vac_{cf}(\tau)(F) = \tau(F)$ iff $E \in {}^{\infty}cf(F)$ so τ is unique wrt. non-emtpy extensions. For the last case we examine the empty set. It holds that $F^{\emptyset} = F$ and by the second case there is no nonempty $E \in \tau(F)$ iff there is no such E in ${}^{\infty}cf(F)$. So $\tau(F) \subseteq \{\emptyset\}$ iff ${}^{\infty}cf(F) \subseteq \{\emptyset\}$. Therefore $\emptyset \in \tau(F)$ iff $\emptyset \in {}^{\infty}cf(F)$, and in that case $\tau(F) = {}^{\infty}cf(F) = \emptyset$ follows. □

D Omitted Proofs of Section 6

In order to prove the following completeness statements we refer to the $\Sigma_i SAT$-problem.(Dvorák and Dunne 2018)

Proposition 36. *Let $\phi = \exists X_{n-1} \forall X_{n-2} ... Q_0 X_0 : \varphi(X_0, ..., X_{n-1})$ be a quantified boolean formula with $X_i = (x_1^i, ..., x_{m_i}^i)$ a vector of Boolean variables for every i, with alternating quantors, making $Q_0 = \exists$ iff $n-1$ is odd. Then the decision problem of deciding whether any such a QBF of degree n is true is Σ_n^p-complete. The analogous problem of deciding the truth of an n-degree QBF $\phi = \forall X_{n-1} \exists X_{n-2} ... Q_0 X_0 : \varphi(X_0, ..., X_n)$ is Π_n^p-complete.*

Theorem 34. *1. $Ver_{vac_{cf}^n(ad)}$ is Π_n^P-complete.*

2. $Exists_{vac_{cf}^n(ad)}^{\neg\emptyset}$ is $\sum_{n+1}^P$-complete.

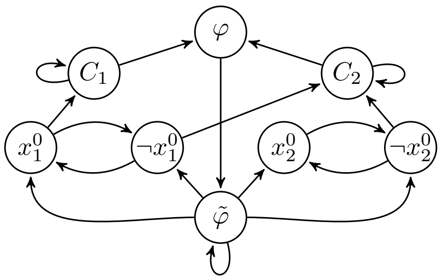
3. $Cred_{vac_{cf}^n(ad)}$ is $\sum_{n+1}^P$-complete.

4. $Skep_{vac_{cf}^n(ad)}$ is Π_{n+1}^P-complete.

5. $Exists_{vac_{cf}^n(ad)}$ is $\sum_{n+1}^P$-complete for $n \neq 0$ even and trivial for n odd.

Proof. 1. Verification is the complement to item 2, i. e. in order to verify a set E is in $vac_{cf}^n(ad)(F)$ we have to show that $vac_{cf}^{n-1}(ad)(F^E)$ contains no non-empty extension. The existence of such an extension is in $\sum_n^P = NP^{\sum_{n-1}^P}$ therefore the nonexistence is in $coNP$ with access to a $\sum_{n-1}^P$-oracle, i. e. in Π_n^P. For completeness the construction from item 2. suffices to prove that a QBF $\phi = \forall X_{n-1} \exists X_{n-2} ... Q_0 X_0 : \varphi(X_{n-1}, ... X_0) = \neg(\exists X_{n-1} \forall X_{n-2} ... Q_0 X_0 : \neg\varphi(X_0, ..., X_{n-1}))$ is true iff F_ϕ contains no non-empty $vac_{cf}^{n-1}(ad)$-extension. In order to make this a verification problem, we can modify

Figure 5: F_ϕ for $\phi = \exists(x_1^0, x_2^0) : x_1^0 \wedge (\neg x_1^0 \vee \neg x_2^0)$



$F_{\neg\phi}$ as follows:

$$A'_\phi = A_{\neg\phi} \cup \{z\}$$
$$R'_\phi = R_{\neg\phi}$$

Then $F'^{\{z\}}_\phi$ is $F_{\neg\phi}$ and $\{z\} \in vac^n_{cf}(ad)(F'_\phi)$ iff ϕ is true.

2. Proof by induction over n, the base case is the preferred semantics $vac^0_{cf}(ad)$.

$Exists^{\neg\emptyset}_{vac^n_{cf}(ad)}$ is in $\sum^P_{n+1}$, because we can guess in NP a conflictfree set and check with a Σ^P_n-oracle that the reduct contains no nonempty $vac^{n-1}_{cf}(ad)$-extension according to the induction hypothesis. For the preferred semantics the problem is in NP (Dvorák and Dunne 2018).
For Σ^P_{n+1}-hardness we will reduce the corresponding QBF-Problem from Prop. 36 of deciding whether the QBF $\phi = \exists X_n \forall X_{n-1} ... Q_0 X_0 : \varphi(X_0, ..., X_n)$ is true to the existence of a non-empty $vac^n_{cf}(ad)$-extension in a suitable AF F_ϕ constructed in polynomial time.

(Base case) For $n = 0$ we use the construction from (Dvorák and Dunne 2018). For $\phi = \exists X_0 : \varphi(X_0)$ with $\varphi(X_0) = \bigwedge_{j=0}^{l} C_j$ in CNF the corresponding AF $F_\phi = (A, R)$ is given by:

$$\begin{aligned}
A_\phi = \{ & x_1^0, \neg x_1^0, ..., \neg x_{m_1}^0 \quad \text{(x-arguments)}\\
& C_0, ..., C_l \quad \text{(clause-arguments)}\\
& \varphi, \tilde{\varphi}\} \quad \text{(phi-arguments)}\\
R_\phi = \{ & (x_i^0, \neg x_i^0), (\neg x_i^0, x_i^0)\ \forall 1 \le i \le m_1 \quad \text{(contradictions)}\\
& (\ddot{x}_i^0, C_j)\ if\ literal\ \ddot{x}_i^0 \in C_j \quad \text{(clause-satisfaction)}\\
& (C_j, \varphi)\ \forall j \quad \text{(φ-satisfaction)}\\
& (C_j, C_j), (\tilde{\varphi}, \tilde{\varphi}) \quad \text{(self-attacks)}\\
& (\varphi, \tilde{\varphi}), (\tilde{\varphi}, \ddot{x}_i^0)\ \forall i\} \quad \text{(link φ with x-arguments)}
\end{aligned}$$

We will denote by $\ddot{x}_i$ anything which may apply to both arguments x_i and $\neg x_i$ and by $\dot{x}_i$ the case where we choose exactly one of the two. Now if and only if a satisfying assignment to the variable $X = (x_1^0, ..., x_{m_1}^0)$ exists we get a non-empty preferred extension of the form $E = \{\dot{x}_1^0, ..., \dot{x}_{m_1}^0, \varphi\}$ such that φ defends the X-arguments from $\tilde{\varphi}$ and the correct selection among the x-arguments defends φ from all clause-arguments. Note that a preferred extension will in this case always contain either x_i or $\neg x_i$ for all i, because otherwise $\{x_i\} \in ad(F^E_\phi)$ and therefore $E \notin pr(F_\phi) = vac_{ad}(ad)(F_\phi)$ by (Thimm 2023). Since the clause-arguments and $\tilde{\varphi}$ are self-attackers, all preferred extensions, if they exist, are of the same type as E.

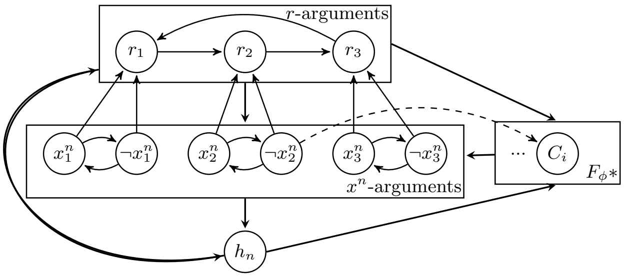
(Induction Step) Let $\phi = \exists X_n \forall X_{n-1} ... Q_0 X_0 : \varphi(X_0, ..., X_n)$ be given. Wlog. we assume that for each $X_i = (x_1^i, ..., x_{m_i}^i)$ the number of variables m_i is odd.[1] First, observe that the QBF ϕ is true iff there exists an assignment $X \in \{0,1\}^n$ to $X_n = (x_1^n, ..., x_{m_n}^n)$ such that the QBF $\phi' = \neg(\forall X_{n-1} ... Q_0 X_0 : \varphi(X_0, ..., X_{n-1}, X)) = \exists X_{n-1} ... Q_0 X_0 : \varphi'(X_1, ..., X_{n-1})$ is false (all quantors are inverted due to negation), where $\varphi'(X_1, ..., X_{n-1}) = \varphi(X_0, ..., X_{n-1}, X)$ is the QBF of degree n where the assignment X is substituted for X_n. By the induction hypothesis there exists an AF $F_{\phi'}$ such that ϕ' is false iff $F_{\phi'}$ contains no non-empty $vac^{n-1}_{cf}(ad)$-extension. We now construct the AF F_ϕ as follows: Let $F_{\phi*}$ be the AF constructed according to the induction hypothesis for the $n-1$-QBF $\phi* = \exists X_{n-1} ... Q_0 X_0 : \neg\varphi(X_0, ..., X_{n-1}, X_n)$, such that all clauses from $\neg\varphi$ are included and the Variable X_n is treated as a free variable and not represented in the construction (it may, for instance, happen, that some clause-arguments are unattacked because of that). F_ϕ is defined by adding the following arguments and attacks to $F_\phi*$.

$$\begin{aligned}
A_\phi = A_{\phi*} \cup & \\
& \{x_1^n, \neg x_1^n, ..., \neg x_{m_n}^n \quad \text{(x-arguments)}\\
& h_n \quad \text{(support-argument)}\\
& r_1^n, ..., r_{m_n}^n\} \quad \text{(restriction-arguments)}\\
R_\phi = R_{\phi*} \cup & \\
& \{(x_i^n, \neg x_i^n), (\neg x_i^n, x_i^n)\ \forall 1 \le i \le m_n \quad \text{(contradictions)}\\
& (\ddot{x}_i^n, C_j)\ if\ literal\ \ddot{x}_i^n \in C_j \quad \text{(clause-satisfaction)}\\
& (\ddot{x}_i^n, r_i^n)\ \forall i \quad \text{(x-selection)}\\
& (r_{m_n}^n, r_1^n), (r_i^n, r_{i+1}^n)\ \forall 1 \le i < m_n \quad \text{(restriction-cycle)}\\
& (r_i^n, \ddot{x}_j^n), (h_n, r_i^n), (r_i^n, h_n), (r_i^n, a)\ \forall i, j \le m_n, a \in A_{\phi*}\\
& \quad \text{(restriction-attacks)}\\
& (\ddot{x}_i^n, h_n), (h_n, a), (a, \ddot{x}_i^n)\ \forall i, a \in A_{\phi*}\ \}\\
& \quad \text{(support-cycle)}
\end{aligned}$$

To proceed, we will first prove the following lemma.

[1]If m_i is even, we simply add an additional variable $x^i_{m_i+1}$ (which does not occur in φ). The resulting formula is semantically equivalent to the original one.

Figure 6: F_ϕ for $X_n = (x_1^n, x_2^n, x_3^n)$



Lemma 37. *If some $E \in vac^n_{cf}(ad)(F_\phi)$, $E \neq \emptyset$ exists, then it has the form $E = \{\dot{x}^n_1, ..., \dot{x}^n_{m_n}\}$ with either $x^n_i \in E$ or $\neg x^n_i \in E$ for all i.*

Proof. Observe that $cf(F_\phi) = cf(F_{\phi*}) \cup \{h_n\} \cup cf(F_\phi|_{\{r^n_1,...,r^n_{m_n}\}}) \cup cf(F_\phi|_{\{\ddot{x}^n_1,...,\ddot{x}^n_{m_n}\}})$.

For any $E \in cf(F_{\phi*})$ the set $\{h\}$ is a $vac^{n-1}_{cf}(ad)$-extension of the reduct F^E_ϕ since h is not attacked by E and attacks everything apart from the x^n-arguments, which are already attacked by E. Thus $F_\phi^{E^{\{h\}}}$ is empty and $E \notin vac^n_{cf}(ad)(F_\phi)$. Since $\{h\}$ attacks everything apart from the x^n-arguments, $\{x^n_1, ..., x^n_{m_n}\}$ is a $vac^{n-1}_{cf}(ad)$-extension of $F^{\{h\}}$. Since the r-arguments form an odd cycle for any conflictfree subset $R \subseteq \{r^n_1, ..., r^n_{m_n}\}$ a non-empty naive set of arguments R' remains in F^R. The r-arguments attack everything else, so $R' \in vac^{n-1}_{cf}(ad)(F^R)$. Last, for a set of x^n-arguments X which does not contain $x^n_i \in E$ or $\neg x^n_i \in E$ for each i a non-empty set of r-arguments will remain in F^X and be a $vac^{n-1}_{cf}(ad)$-extension there. $\square$

By Lemma 37 any non-empty $vac^n_{cf}(ad)$-extension E can be translated in an assignment $X \in \{0,1\}^n$ to $X_n = (x^n_1, ..., x^n_{m_n})$. Observe that for any such E we have $F^E_\phi = F'_\phi$ where $\phi' = \neg(\forall X_{n-1}...Q_0X_0 : \varphi(X_0, ..., X_{n-1}, X)) = \exists X_{n-1}...Q_0X_0 : \varphi'(X_1, ..., X_{n-1})$ as explained above, because the r-arguments and h_n are attacked by the x^n-arguments. Now we can argue as follows:

($\Rightarrow$) Suppose ϕ is true, then an assignment X exists such that ϕ' is false and by the induction hypothesis $F_{\phi'}$ contains no non-empty $vac^{n-1}_{cf}(ad)$-extension. For the corresponding set of x^n-arguments E we therefore have $vac^{n-1}_{cf}(ad)(F^E_\phi) \subseteq \{\emptyset\}$ so E is a non-empty $vac^n_{cf}(ad)$-extension of F_ϕ.

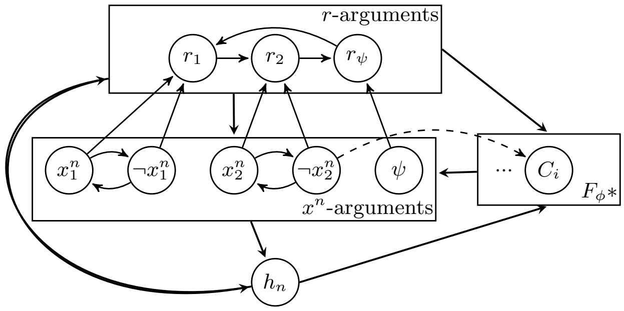
($\Leftarrow$) Suppose ϕ is false, then for all assignments X the corresponding ϕ' is true, so by the induction hypothesis $F_{\phi'}$ contains a non-empty $vac^{n-1}_{cf}(ad)$-extension and the corresponding set of x^n-arguments E is not an $vac^n_{cf}(ad)$-extension of F_ϕ.

3. We use a variation of the construction from item 2, wlog. assume m_n even instead (only m_n, for $i \neq n$ let m_i odd) and add to F_ϕ the following to construct F'_ϕ:

$$A'_\phi = A_\phi \cup \{\psi, r_\psi\}$$
$$R'_\phi = (R_\phi \setminus \{(r^n_{m_n}, r^n_1)\}) \cup \{(\psi, r_\psi), (r_{m_n}, r_\psi), (r_\psi, r^n_1), (r_\psi, h_n), (r_\psi, a) \forall a \in A_\phi*\}$$

Then the $r-arguments$ form an odd cycle and Lemma 37 applies and assures that ψ is part of any non-empty extension of F'_ϕ. So ψ will be accepted iff an assignment to the x^n-arguments exists such that the corresponding ϕ' is false, i. e. iff phi is true.

Figure 7: F'_ϕ for $X_n = (x^n_1, x^n_2)$



4. An even simpler variation of the construction from item 2 can be used here. For a given QBF ϕ let $F'_\phi = (A_\phi \cup \{\psi\}, R_\phi \cup \{(\psi, a), (a, \psi) \mid a \in A_\phi\})$. Then $F'^{\{\psi\}}_\phi$ is empty, so $\{\psi\}$ is a $vac^n_{cf}(ad)$-extension of F'_ϕ due to Prop 23. Since $\{\psi\}$ is also a $vac^{n-1}_{cf}(ad)$-extension, the empty set is not a $vac^n_{cf}(ad)$-extension. Now if ϕ is true, a set E of x^n-arguments exists which attacks ψ and is a $vac^n_{cf}(ad)$-extension of F'_ϕ according to item 2.So ψ is skeptically accepted if and only if ϕ is false.

5. If $n = 2k + 1$ is odd, one of two cases applies: I) no (nonempty) set $E \in vac^{2k}_{cf}(ad)(F)$ exists, then $\emptyset \in vac^{2k+1}_{cf}(ad)(F)$.

II) $E \in vac^{2k}_{cf}(ad)(F)$ exists, then by Theorem 21 this E also satisfies $E \in vac^{2k+1}_{cf}(ad)$. Therefore the problem is trivial for odd n.

For n even item 2 tells us checking for a non-empty set is in $\sum^p_{n+1}$. The empty set is an $vac^n_{cf}(ad)$-extension if $vac^{n-1}_{cf}(ad)(F) \subseteq \{\emptyset\}$, which is in Π^p_n according to item 1. For completeness we prove by induction that for any QBF ϕ of degree $2k + 1$, $k \in \mathbb{N} \setminus \{0\}$ the empty set is not a $vac^n_{cf}(ad)$-extension of F_ϕ from item 2. For $k = 1$ ϕ is of the form $\exists X_2 \forall X_1 \exists X_0 : \varphi(X_2, X_1, X_0)$ with F_ϕ as specified in item 2. Then $E = \{x^2_1, ..., x^2_{m_2}\}$ is a $vac^1_{cf}(ad)$-extension of F_ϕ because in F^E_ϕ the odd cycles between the x^1-arguments, h^1 and the rest of the AF as well as the odd cycle of the r^1-arguments ensure no non-empty preferred extension exists. Thus, the empty set cannot be a $vac^2_{cf}(ad)$-extension of F_ϕ. Our induction hypothesis now is that for $k-1$ and any QBF ϕ of degree $2k-1$ it holds that $\{x_1^{2k-2}, ..., x_{m_{2k-2}}^{2k-2}\} \in vac_{cf}^{2k-3}(ad)(F_\phi)$. Let ϕ be a QBF of degree $2k+1$ and F_ϕ the resp. AF from item 2. Then according to Lemma 37 the only possible $vac_{cf}^{2k-2}(ad)$-extensions of $F_\phi^{\{x_1^{2k}, ..., x_{m_{2k}}^{2k}\}}$ are of the type $E = \{\dot{x}_1^{2k-1}, ..., \dot{x}_{m_{2k-1}}^{2k-1}\}$. For each such E a QBF ϕ' of degree $2k-1$ exists, such that $F_\phi^{\{x_1^{2k}, ..., x_{m_{2k}}^{2k}\}^E}$ is isomorphic to $F_{\phi'}$. By the induction hypothesis any such $F_{\phi'}$ always contains a non-empty $vac_{cf}^{2k-3}(ad)$-extension, so $E \notin vac_{cf}^{2k-2}(ad)(F_\phi^{\{x_1^{2k}, ..., x_{m_{2k}}^{2k}\}})$ and therefore $\{x_1^{2k}, ..., x_{m_{2k}}^{2k}\} \in vac_{cf}^{2k-1}(ad)(F_\phi)$. $\square$