A Ranking Semantics for Abstract Argumentation based on Serialisability

Proofs of technical results

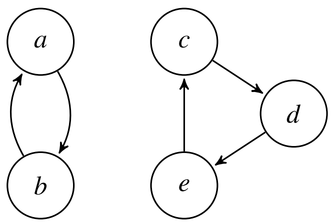
Lydia BLMEL and Matthias THIMM
Artificial Intelligence Group, University of Hagen, Germany

Proposition 1. *A ranking semantics* τ *satisfies self-contradiction iff it satisfies* na-compatibility.

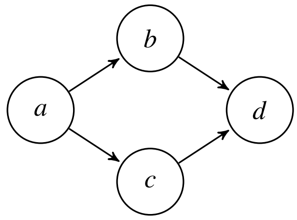
Proof. Let a be a self-attacker and b not, then $\{b\}$ is conflict-free and therefore some $E \in \mathsf{na}(\mathsf{AF})$ exists with $b \in E$. Since $\{a\}$ is not conflict-free no such E exists for a. So the set of credulously accepted arguments by naive semantics is exactly the set of all arguments not attacking themselves. □

Proposition 2. *Let* τ *be a ranking semantics satisfying* adm-compatibility*. Then* τ *does not satisfy any of the following four principles:* strict counter-transitivity, counter-transitivity, cardinality precedence*, and* quality precedence*.*

Proof. (strict counter-transitivity&counter-transitivity) For the AF below adm-compatibility would imply $a \succ_\tau c$ and $b \succ_\tau d$ but under (strict) counter-transitivity not both can hold.



(cardinality precedence) In the AF below b is not acceptable and has one attacker, while the acceptable d has two.

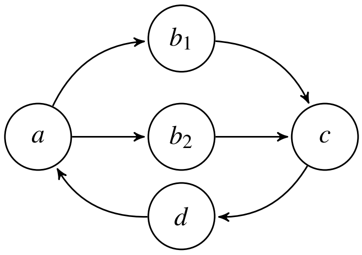


(quality precedence) consider the counterexample for strict counter-transitivity

□

Proposition 3. Strong σ-support *does not imply* weak σ-support *and* weak σ-support *does not imply* strong σ-support.

Proof. Let τ be the serialisation ranking on all $\mathsf{AF} \in \mathfrak{AF}$ except for those isomorphic to the following:



and let the ranking on the above AF under τ be $a \simeq_\tau d \succ_\tau c \simeq_\tau b_1 \simeq_\tau b_2$. Since a and c weakly σ-support each other but neither σ-supports the other strongly τ violates *weak* adm*-support* while according to Theorem 1 τ satisfies *strong* adm*-support*. For the other direction the trivial ranking with $a \simeq_\tau b$ for any two arguments a, b of any AF works as a counterexample. □

Corollary 1. *Let* τ *be a ranking semantics satisfying* weak σ-support, $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ *an AF. If* $|\sigma(\mathsf{AF})| = 1$ *then for all credulously accepted* $a, b \in \mathsf{A}$, $a \simeq_{\tau(AF)} b$.

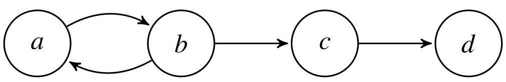
Proof. Let E be the single σ-extension of an AF $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$. Then for any two arguments $a, b \in E$ it holds that a is in every σ-extension containing b and vice versa(because there is only one), so if τ satisfies *weak* σ*-support* $a \succeq_\tau b$ and $b \succeq_\tau a$, which is equivalent to $a \simeq_\tau b$. □

Corollary 2. *Let* τ *be a ranking semantics satisfying* weak σ-support, $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ *an AF. For all skeptically accepted* $a, b \in \mathsf{A}$, $a \simeq_{\tau(AF)} b$.

Proof. For a given $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ let $a, b \in E$ be skeptically accepted arguments then for any $E \in \sigma(\mathsf{AF})$ it holds that $a, b \in E$ so b is an element of every extension containing a and vice versa. Therefore $a \simeq_\tau b$ must hold for any ranking semantics τ which satisfies *weak* σ*-support*. □

Proposition 4. *Let* τ *be a ranking semantics satisfying* strong adm-support*. Then* τ *does not satisfy any of the following four principles:* strict counter-transitivity*,* counter-transitivity*,* cardinality precedence *and* quality precedence*.*

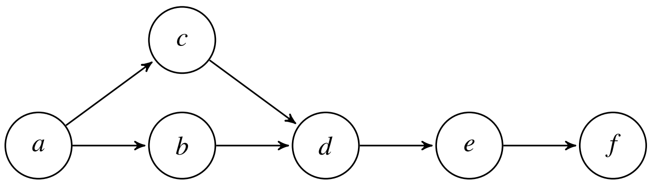
Proof. Suppose some ranking semantics τ satisfies *strong* adm*-support*.



((strict) counter-transitivity) Then both $a \succ_\tau c$ and $b \succ_\tau d$ hold. But under Strict Counter-Transitivity $a \succ_\tau c$ implies $d \succ_\tau b$, so τ cannot satisfy both. Under *counter-transitivity* at least $d \succeq_\tau b$ would have to hold, resulting in the same contradiction.

(quality precedence) Follows from the Counterexample for strict counter-transitivity directly above

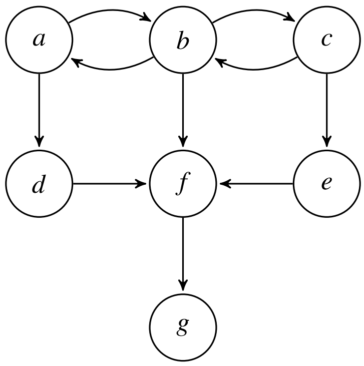
(cardinality precedence) In the AF below d strongly adm-supports f so $d \succ_\tau f$ but d has more attackers, so by cardinality precedence $f \succ_\tau d$. Contradiction.



□

Proposition 5. *Let τ be a ranking semantics satisfying* weak adm-support. *Then τ does not satisfy* strict counter-transitivity *or* cardinality precedence.

Proof. Suppose some ranking semantics τ satisfies *weak adm-support*.



((strict) counter-transitivity) b weakly adm-supports g, so $b \succeq_\tau g$ and a, c weakly adm-support f so $a \succeq_\tau f$ and $c \succeq_\tau f$. Since both attackers of b are ranked higher or equal to the only attacker f of g by *strict counter-transitivity* it follows that $g \succ_\tau b$. Contradiction.

(cardinality precedence) As b has more attackers than g the counterexample for strict counter-transitivity applies.

□

Theorem 1. $\succeq_{ser}$ *satisfies* adm-compatibility *and both* strong *and* weak adm-support.

Proof. Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R}) \in \mathfrak{AF}$, $a, b \in \mathsf{A}$. *adm-compatibility* follows directly from the definition, since $\mathsf{ser}_{\mathsf{AF}}(b) = \infty > n = \mathsf{ser}_{\mathsf{AF}}(a)$ for any non-admissible argument b and admissible argument a.

For *weak adm-support* suppose $a \in E$ for any admissible E containing b, then any admissible set constructed in the minimum number of steps $n = \mathsf{ser}_{\mathsf{AF}}(b)$ contains a as well, thus $\mathsf{ser}_{\mathsf{AF}}(a) \leq \mathsf{ser}_{\mathsf{AF}}(b)$.

For *strong adm-support*, suppose a, b admissible such that for each $E \in \mathsf{adm}(\mathsf{AF})$ containing b an admissible subset E' containing a but not b exists. Let $(S_1, ..., S_n)$ be the serialisation sequence of such an E with the minimal number of serialisation steps $n = \mathsf{ser}_{\mathsf{AF}}(b)$. Then $b \in S_n$. If $a \in S_1 \cup ... \cup S_{n-1}$ we are done. Now suppose $a \in S_n$, then some attacker a' of a in $\mathsf{AF}^{S_1 \cup ... \cup S_{n-1}}$ exists. By definition of the reduct a' is not attacked by $S_1 \cup ... \cup S_{n-1}$, nor by any subset of this set, so some $c \in S_n$ must exist which attacks a'. Since S_n is an initial set, this means $\{a, c\}$ is not admissible in $\mathsf{AF}^{S_1 \cup ... \cup S_{n-1}}$, so some attacker of c must exist that is not attacked by neither c, a nor $S_1 \cup ... \cup S_{n-1}$. This argument can be repeated until b itself has to be added in order to achieve admissibility. Therefore no admissible proper subset of E containing a exists. Contradiction. ☐

Proposition 7. $\succeq_{ser}$ *satisfies* directionality.

Proof. For a given AF $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ suppose $a, b, x \in \mathsf{A}$, $(a, b) \notin \mathsf{R}$ and no path from b to x exists. Let $\mathsf{AF}' = (\mathsf{A}, \mathsf{R} \cup \{(a, b)\})$. We will prove the statement by showing that $\mathsf{ser}_{\mathsf{AF}}(x) = \mathsf{ser}'_{\mathsf{AF}}(x)$. The classic admissibility semantics satisfies directionality [22], so x is not admissible in AF iff it is not admissible in AF', in both cases we have $\mathsf{ser}_{\mathsf{AF}}(x) = \infty = \mathsf{ser}'_{\mathsf{AF}}$.

Suppose $\mathsf{ser}_{\mathsf{AF}}(x) \neq \infty$, then x is admissible and a minimal serialisation sequence $(S_1, ..., S_n)$ with $n = \mathsf{ser}_{\mathsf{AF}}(x)$ for an admissible S containing x exists. By the following lemma all arguments of such S_i have a path to x which means in return there is no path from b to those sets, neither in AF nor in any of its reducts. Prop. 1 of [19] states that initial sets are constrained to a single scc. The reduct does not add any attacks, so arguments which are not part of the same scc in AF are not in the same scc in AF^S, too. By the directionality of the admissible semantics it follows for any i, $1 \leq i \leq n$, that S_i is admissible in $\mathsf{AF}^{S_1 \cup ... \cup S_{i-1}}$ iff it is admissible in $\mathsf{AF}'^{S_1 \cup ... \cup S_{i-1}}$. Since there is also no path from b to any admissible set contained in the SCC $C_i \in SCC(\mathsf{AF})$ which contains S_i this also holds for any admissible set $T \in \mathsf{adm}(C_i)$, so S_i is still an initial set in $\mathsf{AF}'^{S_1 \cup ... \cup S_{i-1}}$ by Prop. 2 of [19]. Therefore $S_1, ..., S_n$ is a serialisation sequence for x in AF', too, so $\mathsf{ser}_{\mathsf{AF}}(x) \geq \mathsf{ser}'_{\mathsf{AF}}(x)$.

Now suppose a shorter serialisation sequence $S'_1, ..., S'_m$ for some admissible S' containing x existed in AF'. Then by the same argumentation as for S this sequence would be a serialisation sequence for x in the original framework AF, so S was not minimal to begin with. Contradiction, therefore $\mathsf{ser}_{\mathsf{AF}}(x) = \mathsf{ser}'_{\mathsf{AF}}$. ☐

Lemma 1. *Let* $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ *be an AF,* $x \in \mathsf{A}$ *an admissible argument and let* $S \in adm(\mathsf{AF})$ *be an admissible set containing* x.[2] *Let* $S_0 = \emptyset, S_1, ..., S_n$ *be a minimal serialisation sequence for* S*, then* $x \in S_n$ *and for any* S_i *and any* $y \in S_i$ *a path from* y *to* x *exists.*

Proof. We will show this recursively. If $n = 1$ then this holds because any initial set lies in an scc. Now let $n > 1$ and S as specified in the Lemma. By Th. 2 [19] $S' = S_1 \cup ... \cup S_{n-1}$ is admissible. Now if $x \notin S_n$ it would be an element of S' and S would not be a minimal admissible set containing x, therefore $x \in S_n$. We know S_n is admissible in $\mathsf{AF}^{S'}$. If there was no path from S_{n-1} to S_n, then $S_n^- \cap \mathsf{AF}^{S'} = S_n^- \cap \mathsf{AF}^{S_1 \cup ... \cup S_{n-2}}$ and thus S_n would be admissible in $\mathsf{AF}^{S_1 \cup ... \cup S_{n-2}}$, which means a smaller admissible set with a shorter serialisation containing x would exist. Contradiction. But since S_{n-1} and S_n both

[2] Choose an S with the least serialisation steps among admissible extensions containing x

lie within their resp. sccs this means any element of S_{n-1} has a path to x. Recursively, if for any $i \leq n$ no path from S_i to any $S_j,\ j > i$ existed, the set $S" = S_{i+1} \cup \ldots \cup S_n$ would already be admissible in $\mathsf{AF}^{S_1 \cup \ldots \cup S_{i-1}}$ and S_i would not be part of a minimal admissible set for x. Therefore a path from S_i to some S_j must exist and by recursion S_j already has a path to x. Because of the scc-containment any argument of S_i therefore has a path to x. □

Proposition 8. $\succeq_{ser}$ *satisfies* abstraction*,* independence*,* totality*,* non-attacked equivalence*, and* attack vs full defense*. All other principles from Def. 3 are not satisfied.*

Proof. (abstraction) Admissible semantics is invariant under isomorphisms, that is $\{\gamma(E) \mid E \in adm(\mathsf{AF})\} = adm(\gamma(\mathsf{AF}))$ for any bijective, attack-preserving function $\gamma : \mathfrak{AF} \to \mathfrak{AF}$. If the admissible extensions are preserved, the initial sets are preserved and since the attack relation is preserved, $\gamma(S^-) = (\gamma(S))^-$ for any $S \in \mathsf{IS}(\mathsf{AF})$, the reduct is preserved as well. Thus, if $\mathscr{S} = (S_1, \ldots, S_n)$ is a serialization sequence in AF then $\gamma(\mathscr{S}) = (\gamma(S_1), \ldots, \gamma(S_n))$ is a serialization sequence in $\gamma(\mathsf{AF})$. Therefore $\mathsf{ser}_{\mathsf{AF}}(a) = \mathsf{ser}_{\mathsf{AF}}(\gamma(a))$ holds for any argument and thus $\succeq_{\mathsf{ser}}$ satisfies abstraction.

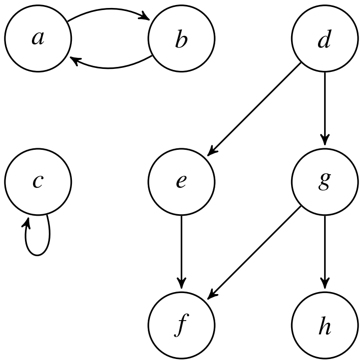
(independence) The admissible semantics satisfies directionality [7]. Consequently an argument $a \in C$ with $C \in cc(\mathsf{AF})$ a connected comp. is admissible in $\mathsf{AF}|_C$ iff a is admissible in AF, so independence for non-admissible arguments is trivial for $\succeq_{\mathsf{ser}}$. Now let a be admissible, then by directionality any serialisation sequence for a in C is a serialisation sequence in AF, because by directionality the initial sets of C are minimal admissible in AF, too (and this holds for any reduct, because the cc-status is preserved). In [19] it is shown that on the other hand any $S \in \mathsf{IS}(\mathsf{AF})$ with $S \not\subseteq C$ is entirely a member of $A \setminus C$, so a serialisation step with such an S would change nothing wrt. C in the reduct, such a step can be added in any sequence for a but can never make one shorter. Therefore the minimum sequence always lies within C, so for any $a, b \in C$ we have $\mathsf{ser}_C(a) \leq \mathsf{ser}_C(b)$ iff $\mathsf{ser}_{\mathsf{AF}}(a) \leq \mathsf{ser}_{\mathsf{AF}}(b)$.

(totality) per definition

(non-attacked equivalence) By [19] all unattacked arguments are initial sets themselves, so for any two unattacked arguments a, b serialisation needs only one step, thus $\mathsf{ser}_{\mathsf{AF}}(a) = \mathsf{ser}_{\mathsf{AF}}(b) = 1$.

(attack vs full defense) Arguments with only defense roots in an acyclic AF are admissible because they are part of the grounded extension, so $\mathsf{ser}_{\mathsf{AF}}(a) \in \mathbb{N}$ for any fully-defended argument a. An argument b attacked by a non-attacked argument on the other hand always satisfies $\mathsf{ser}_{\mathsf{AF}}(b) = \infty$ so it is always ranked lower than a.

The other principles are contradicted in the following counterexamples.



(void Precedence) $\mathsf{ser}_{\mathsf{AF}}(a) = \mathsf{ser}_{\mathsf{AF}}(d) = 1$, so the unattacked $d \not\succ_{\mathsf{ser}} a$.

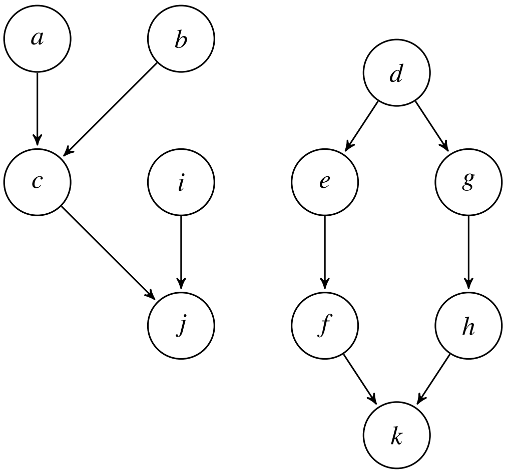
(self-contradiction) $\mathsf{ser}_{\mathsf{AF}}(e) = \mathsf{ser}_{\mathsf{AF}}(c) = \infty$

(cardinality precedence) $\mathsf{ser}_{\mathsf{AF}}(f) = \mathsf{ser}_{\mathsf{AF}}(h) = 2$, but $|f^-| = 2 > 1 = |h^-|$

(quality precedence) $\mathsf{ser}_{\mathsf{AF}}(e) = \mathsf{ser}_{\mathsf{AF}}(c) = \infty$, but $\mathsf{ser}_{\mathsf{AF}}(c^-) = \mathsf{ser}_{\mathsf{AF}}(c) = \infty > 1 = \mathsf{ser}_{\mathsf{AF}}(d)$

((strict) counter-transitivity) $\mathsf{ser}_{\mathsf{AF}}(a) = 1 < 2 = \mathsf{ser}_{\mathsf{AF}}(h)$, so a is more acceptable than h while at the same time the attacker of a is more acceptable than the one of h, that is $\mathsf{ser}_{\mathsf{AF}}(a^-) = \mathsf{ser}_{\mathsf{AF}}(b) = 1 < \infty = \mathsf{ser}_{\mathsf{AF}}(g) = \mathsf{ser}_{\mathsf{AF}}(h^-)$, and both arguments have only one attacker, by reverse implication this counters *strict counter-transitivity*, too

For the two defense principles, see the counterexample below.



(defense precedence) Consider only the black arguments. $\mathsf{ser}_{\mathsf{AF}}(j) = \mathsf{ser}_{\mathsf{AF}}(c) = \infty$ and $|j^-| = |c^-| = 2$ but $(j^-)^-$ is not empty while $(c^-)^- = \emptyset$

(distributed defense precedence) Consider only the black arguments. $\mathsf{ser}_{\mathsf{AF}}(j) = \mathsf{ser}_{\mathsf{AF}}(k) = \infty$ while $|j^-| = |k^-| = 2 = |(j^-)^-| = |(k^-)^-|$ and each defender attacks exactly 1 attacker, with attacker i of j staying unattacked.

$\square$