Serialisable Semantics for Abstract Argumentation

Proofs of technical results

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Theorem 1. *Let σ be any semantics.*

- *If σ satisfies serialisability then it satisfies conflict-freeness.*
- *If σ satisfies serialisability then it satisfies admissibility.*
- *If σ satisfies serialisability then it satisfies modularization.*
- *Directionality does not imply serialisability and vice versa.*
- *SCC-recursiveness does not imply serialisability and vice versa.*

Proof. We proof each statement separately as follows:

- *If a semantics σ satisfies serialisability then it satisfies conflict-freeness.*
  Follows from the fact that every serialisable semantics is admissible, since conflict-freeness is subsumed by admissibility.
- *If a semantics σ satisfies serialisability then it satisfies admissibility.*
  Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an argumentation framework, σ is a semantics serialisable via α_σ and β_σ and $E \in \sigma(\mathsf{AF})$ is a σ-extension of AF. Assume that E is not admissible. We know that there exists some admissible set $E_1 \in \mathsf{IS}(\mathsf{AF})$ such that $E_1 \subseteq E$, i.e. E_1 has been selected first from α_σ during the construction of E. In every subsequent step $i = 2, \ldots, n$ of the transition system until termination we only select initial sets E_i of the respective reduct of AF. That means, there can be no conflict between any E_i. Since every E_i is admissible in the respective reduct per definition, it follows that $E = \bigcup_{i=1,\ldots,n} E_i$ is admissible as well and, thus, every serialisable semantics satisfies admissibility.
- *If a semantics σ satisfies serialisability then it satisfies modularization.*
  Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an argumentation framework, $E_1, E_2 \subseteq \mathsf{A}$ are sets of arguments and σ is a semantics serialisable via α_σ and β_σ. Assume we have that E_1 is a σ-extension of AF, i.e., $E_1 \in \sigma(\mathsf{AF})$ and we also have that $E_2 \in \sigma(\mathsf{AF}^{E_1})$. Then, since E_1 can be constructed via the transition system, we can also write E_1 as a sequence of initial sets $S_1 \cup \cdots \cup S_n$ such that $S_1 \in \mathsf{IS}(\mathsf{AF})$ and for all $i = 2, \ldots, n$ we have that $S_i \in \mathsf{IS}(\mathsf{AF}^{S_1 \cup \cdots \cup S_{i-1}})$. Similarly, we can write E_2 as a sequence of initial sets $T_1 \cup \cdots \cup T_m$ such that $T_1 \in \mathsf{IS}(\mathsf{AF}^{E_1})$ and for all $j = 1, \ldots, m$ we have that $T_j \in$

$\mathsf{IS}(\mathsf{AF}^{{E_1}^{T_1\cup\cdots\cup T_{j-1}}})$. Recall, that $\mathsf{AF}^{E'^{E''}}$ is the same as $\mathsf{AF}^{E'\cup E''}$ for any $E', E'' \subseteq \mathsf{A}$. Since $E_1 = S_1 \cup \cdots \cup S_n$ and $T_1 \in \mathsf{IS}(\mathsf{AF}^{E_1})$, it follows that $T_1 \in \mathsf{IS}(\mathsf{AF}^{S_1\cup\cdots\cup S_n})$. Therefore, $S_1 \cup \cdots \cup S_n \cup T_1 \cup \cdots \cup T_m$ also represents a sequence of initial sets that can be constructed by the transition system for α_σ and β_σ. So, $E_1 \cup E_2 = S_1 \cup \cdots \cup S_n \cup T_1 \cup \cdots \cup T_m$ is a σ-extension of AF and thus σ satisfies modularization.

- *Directionality does not imply serialisability and vice versa.*
  The ideal semantics is directional, but not serialisable. The stable semantics is serialisable, but not directional.
- *SCC-recursiveness does not imply serialisability and vice versa.*
  The CF2 semantics is SCC-recursive, but not serialisable. The unchallenged semantics is serialisable, but not SCC-recursive.

□

Theorem 2. *The adm, co, gr, pr and sa semantics are $\alpha\beta$-closed, while the st semantics is not, w.r.t. the functions defined in [6].*

Proof. **Admissible Semantics** The transition system for the admissible semantics with α_{ad} and $\beta_{ad} = 1$ trivially terminates for every path, since β_{ad} is per definition always true.

Complete Semantics The complete semantics can be serialised with the selection function α_{ad} and the termination function β_{co}. Let $\mathsf{AF}_0 = (\mathsf{A}, \mathsf{R})$ be an argumentation framework and $S \subseteq \mathsf{A}$ is a set of arguments. Consider the state (AF_1, S_1) with $(\mathsf{AF}_0, \emptyset) \rightsquigarrow^{\alpha_{ad}} (\mathsf{AF}_1, S_1)$. Assume that (AF_1, S_1) is not terminal, i.e., $\beta_{co}(\mathsf{AF}_1, S_1) = 0$. However, then is follows that $\mathsf{IS}^{\not\leftarrow}(\mathsf{AF}_1) \neq \emptyset$. Then, there is at least one initial set S' which is selectable wrt. α_{ad}. Thus, there exists some state (AF_2, S_2) such that $(\mathsf{AF}_1, S_1) \rightsquigarrow^{\alpha_{ad}} (\mathsf{AF}_2, S_2)$. Trivially, we also have that $\beta_{co}((\emptyset, \emptyset), S_1) = 1$. It follows, that for every state we will eventually reach a followup state (AF_n, S_n) where the argumentation framework AF_n is empty or there exists no initial set that can be selected. In both cases the termination function β_{co} is true and, thus, the complete semantics is $\alpha\beta$-closed.

Grounded Semantics The complete semantics can be serialised with the selection function α_{gr} and the termination function β_{co}. Let $\mathsf{AF}_0 = (\mathsf{A}, \mathsf{R})$ be an argumentation framework and $S_1 \subseteq \mathsf{A}$ is a set of arguments. Consider the state (AF_1, S_1) with $(\mathsf{AF}_0, \emptyset) \rightsquigarrow^{\alpha_{gr}} (\mathsf{AF}_1, S_1)$. Assume that (AF_1, S_1) is not terminal, i.e., $\beta_{co}(\mathsf{AF}_1, S_1) = 0$. However, then is follows that $\mathsf{IS}^{\not\leftarrow}(\mathsf{AF}_1) \neq \emptyset$. Then, there is at least one initial set S' which is selectable wrt. α_{gr}. Thus, there exists some state (AF_2, S_2) such that $(\mathsf{AF}_1, S_1) \rightsquigarrow^{\alpha_{gr}} (\mathsf{AF}_2, S_2)$. Trivially, we also have that $\beta_{gr}((\emptyset, \emptyset), S_1) = 1$. It follows, that for every state we will eventually reach a followup state (AF_n, S_n) where the argumentation framework AF_n is empty or there exists no unattacked initial set that can be selected. In both cases the termination function β_{co} is true and, thus, the grounded semantics is $\alpha\beta$-closed.

Preferred Semantics The complete semantics can be serialised with the selection function α_{ad} and the termination function β_{pr}. Let $\mathsf{AF}_0 = (\mathsf{A}, \mathsf{R})$ be an argumentation framework and $S \subseteq \mathsf{A}$ is a set of arguments. Consider the state (AF_1, S_1) with $(\mathsf{AF}_0, \emptyset) \rightsquigarrow^{\alpha_{ad}} (\mathsf{AF}_1, S_1)$. Assume that (AF_1, S_1) is not terminal, i.e., $\beta_{pr}(\mathsf{AF}_1, S_1) = 0$. However, then is follows that $\mathsf{IS}(\mathsf{AF}_1) \neq \emptyset$. Then, there is at least one initial set S' which is selectable wrt. α_{ad}. Thus, there exists some state (AF_2, S_2) such that $(\mathsf{AF}_1, S_1) \rightsquigarrow^{\alpha_{ad}} (\mathsf{AF}_2, S_2)$. Trivially, we also have that $\beta_{pr}((\emptyset, \emptyset), S_1) = 1$. It follows, that for every state we will eventually reach a followup state (AF_n, S_n) where the argumentation framework

AF_n is either empty or there exists no initial set that can be selected. In both cases the termination function β_{pr} is true and, thus, the preferred semantics is $\alpha\beta$-closed.

Strong Admissible Semantics The transition system for the admissible semantics with α_{gr} and β_{ad} trivially terminates for every path, since β_{ad} is per definition always true.

Stable Semantics The complete semantics can be serialised with the selection function α_{ad} and the termination function

$$\beta_{st}(\mathsf{AF},S) = \begin{cases} 1 \text{ if } \mathsf{AF} = (\emptyset,\emptyset) \\ 0 \text{ otherwise} \end{cases}$$

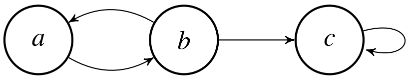

Figure 4. The argumentation framework AF_5.

Consider the argumentation framework AF_5 depicted in Figure 4. The initial sets of AF_5 are $\{a\}$ and $\{b\}$. Assume we select $\{a\}$ in the first transition step, i.e., we transition

$$(\mathsf{AF}_5,\emptyset) \xrightarrow{\{a\}\in\alpha_{ad}(\emptyset,\emptyset,\{\{a\},\{b\}\})} (\mathsf{AF}_5^{\{a\}},\{a\}).$$

Now, $\mathsf{IS}(\mathsf{AF}_5^{\{a\}}) = \emptyset$, therefore no further transition is possible. However, we also have that $\beta_{st}(\mathsf{AF}_5^{\{a\}},\{a\}) = 0$. Thus, this transition path does not lead to a stable extension and therefore the stable semantics is not $\alpha\beta$-closed. $\square$

Theorem 3. *If a semantics σ is serialisable via α_σ and β_σ and is $\alpha_\sigma\beta_\sigma$-closed, then σ satisfies directionality.*

Proof. Let $\mathsf{AF} = (\mathsf{A},\mathsf{R})$ be an argumentation framework and σ is a semantics that is serialisable via α_σ and β_σ and is $\alpha_\sigma\beta_\sigma$-closed.

We have to show, for all $U \in \mathfrak{US}(\mathsf{AF})$ it holds that $\sigma(\mathsf{AF},U) = \sigma(\mathsf{AF}|_U)$ with $\sigma(\mathsf{AF},U) = \{E \cap U \mid E \in \sigma(\mathsf{AF})\}$. For that, we have to show two directions:

1. $\sigma(\mathsf{AF}|_U) \subseteq \sigma(\mathsf{AF},U)$:
   Let $E \in \sigma(\mathsf{AF}|_U)$ be an extension of $\mathsf{AF}|_U$. Then there exists a sequence of initial sets $E_1,\dots,E_n$ such that $E_1 \cup \dots \cup E_n = E$ with $E_1 \in \mathsf{IS}(\mathsf{AF}|_U)$ and $\forall i = 2,\dots,n:$ $E_i \in \mathsf{IS}(\mathsf{AF}|_U^{E_1\cup\dots\cup E_{i-1}})$. As U is an unattacked set of AF and E_1 is admissible in U it follows that E_1 cannot be attacked by arguments outside of U. So E_1 must be admissible in AF as well. It is also necessarily an initial set in AF (otherwise it would not be an initial set in $\mathsf{AF}|_U$). We also have that E_2 is admissible in $\mathsf{AF}|_U^{E_1}$. Again, E_2 cannot be attacked by arguments outside of U, meaning E_2 is clearly either defended by E_1 or itself against all relevant attacks in AF, thus it is also admissible in AF^{E_1}. Therefore, it follows inductively that all E_i are also admissible in the respective reducts $\mathsf{AF}^{E_1\cup\dots\cup E_{i-1}}$ for all $i = 2,\dots,n$. Since σ is $\alpha_\sigma\beta_\sigma$-closed, we know that every transition path has to terminate eventually. Therefore, there must be some σ-extension $E' \supseteq E$ for which $\beta_\sigma(\mathsf{AF}',E') = 1$.

2. $\sigma(\mathsf{AF}, U) \subseteq \sigma(\mathsf{AF}|_U)$:
   Let $E \in \sigma(\mathsf{AF})$ be a σ-extension of AF. Then there exists a sequence of initial sets $E_1, \ldots, E_n$ such that $E_1 \cup \cdots \cup E_n = E$ with $E_1 \in \mathsf{IS}(\mathsf{AF})$ and $\forall i = 2, \ldots, n : E_i \in \mathsf{IS}(\mathsf{AF}^{E_1 \cup \cdots \cup E_{i-1}})$. Recall that in an SCC S, for every pair of arguments $a, b \in S$ there exists an attack path between them. That means, if we have an unattacked set $U \subseteq \mathsf{A}$ and some argument $a \in U$ and a is in the SCC S, then it follows that all arguments $b \in S$ must also be in U. In other words, a S is either completely contained in the unattacked set U no arguments of S is in U at all. Furthermore, every initial set $E_i \in \mathsf{IS}(\mathsf{AF})$ is always contained within some SCC S of AF [6], Proposition 1. It follows, that for every initial set E_i of the sequence we have that either $E_i \subseteq U$ or $E_i \subseteq \mathsf{A} \setminus U$, i. e., every E_i lies either fully in $\mathsf{AF}|_U$ or not at all. Let $j_1, \ldots, j_m$ be the indices of those initial sets in the sequence $E_1, \ldots, E_n$ that are contained in U with $\{j_1, \ldots, j_m \subseteq \{1, \ldots, n\}\}$ and $\forall x \in \{1, \ldots, m\} : j_x < j_{x+1}$. In other words, the initial sets $E_{j_1}, \ldots, E_{j_m}$ are exactly those that are contained in U while all other initial sets E_i of the sequence $E_1, \ldots, E_n$ are not in U.
   We now have to show that $E_{j_1}, \ldots, E_{j_m}$ is a valid transition sequence in $\mathsf{AF}|_U$ and $E' = E_{j_1} \cup \cdots \cup E_{j_m}$ a σ-extension of $\mathsf{AF}|_U$. Every initial set E_{j_x} of the sequence $E_{j_1}, \ldots, E_{j_m}$ is only attacked by arguments $b \in U$, but since it is admissible in AF, it is also admissible in $\mathsf{AF}|_U$ (and therefore also initial). Without loss of generality, consider two initial sets $E_{j_x}, E_{j_{x+1}}$ out of the sequence $E_{j_1}, \ldots, E_{j_m}$. Then, there exists an initial set E_y in $E_1, \ldots, E_n$ such that $j_x = y$. Now, one of two cases applies:

   (a) If $j_{x+1} = y + 1$, then E_{j_x} and $E_{j_{x+1}}$ are adjacent in the original sequence and we have that $E_{j_{x+1}} \in IS(\mathsf{AF}^{E_1 \cup \cdots \cup E_{j_x}})$. Moving to $\mathsf{AF}|_U$, we can ignore all $E_i \notin E_{j_1}, \ldots, E_{j_m}$ in the reduct step, because they are not in U and thus $\mathsf{AF}|_U = \mathsf{AF}|_U^{E_i}$. That means, we also have that $E_{j_{x+1}} \in IS(\mathsf{AF}|_U^{E_{j_1} \cup \cdots \cup E_{j_x}})$, i.e. $E_{j_{x+1}}$ is also an initial set of the corresponding reduct of $\mathsf{AF}|_U$.
   (b) If $j_{x+1} \geq y + 1$, that means there is at least one initial set $E_i \subseteq \mathsf{A} \setminus U$ between them in the original sequence for E. So, before selecting $E_{j_{x+1}}$, we move to the reduct $\mathsf{AF}^{E_1 \cup \cdots \cup E_{j_x} \cup E_i}$ by removing the arguments $E_i \cup E_i^+$. However, since E_i is not contained in U, all arguments that are removed are not in U as well.

   It follows for all x that $E_{j_{x+1}} \in IS(\mathsf{AF}|_U^{E_{j_1} \cup \cdots \cup E_{j_x}})$ in any case. Therefore, $E_{j_1}, \ldots, E_{j_m}$ is a valid transition sequence for $\mathsf{AF}|_U$.
   What remains is to show that $E' = E_{j_1} \cup \cdots \cup E_{j_m}$ is a σ-extension of $\mathsf{AF}|_U$. Assume the opposite is true, i.e, $\beta_\sigma(\mathsf{AF}|_U^{E_{j_1} \cup \cdots \cup E_{j_m}}, E') = 0$. Since σ is $\alpha_0\beta_0$-closed, the sequence $E_{j_1}, \ldots, E_{j_m}$ can be continued in $\mathsf{AF}|_U^{E_{j_1} \cup \cdots \cup E_{j_m}}$. So there must exist an initial set $\hat{E}$ of $\mathsf{AF}|_U^{E_{j_1} \cup \cdots \cup E_{j_m}}$ that is selected by α_0. As before, $\hat{E}$ is then also an initial set in $\mathsf{AF}^{E_1 \cup \cdots \cup E_n}$, and the sequence $E_1, \ldots, E_n$ could be extended in AF as well, contradicting the assumption.
   Therefore, $E' = E \cap U$ must be a σ-extension of $\mathsf{AF}|_U$.

It follows that the $\alpha_\sigma\beta_\sigma$-closure of a serialisable semantics σ implies directionality. □

Theorem 5. *Unchallenged semantics is $\alpha_{uc}\beta_{uc}$-closed.*

Proof. Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an argumentation framework and $S \subseteq \mathsf{A}$ is a set of arguments. Consider the state (AF_1, S_1) obtained by starting in the transition system from the empty set, i.e. $(\mathsf{AF}, \emptyset) \rightsquigarrow^{\alpha_{uc}} (\mathsf{AF}_1, S_1)$. The eventual termination of each state follows quite easily from the definitions of α_{uc} and β_{uc}. Assume $\beta_{uc}(\mathsf{AF}_1, S_1) = 0$, then it follows from the definition that $\mathsf{IS}^{\not\leftarrow}(\mathsf{AF}) \cup \mathsf{IS}^{\not\leftrightarrow}(\mathsf{AF}) \neq \emptyset$. Therefore, we have that $\alpha_{uc}(\mathsf{IS}^{\not\leftarrow}, \mathsf{IS}^{\not\leftrightarrow}, \mathsf{IS}^{\leftrightarrow}) \neq \emptyset$ which means there exists some state (AF_2, S_2) with $\mathsf{AF}_2 = \mathsf{AF}_1^{S'}$ and $S_2 = S_1 \cup S'$ for some $S' \in \mathsf{IS}^{\not\leftarrow}(\mathsf{AF}) \cup \mathsf{IS}^{\not\leftrightarrow}(\mathsf{AF})$. So, whenever β_{uc} is false, there exists per definition a follow-up state.

Now, all we need to show is that there is a finite number of transition steps for every argumentation framework. Per definition, every initial set $S \in \mathsf{IS}(\mathsf{AF})$ is not empty, i.e., $\forall S \in \mathsf{IS}(\mathsf{AF}) : S \neq \emptyset$. Therefore, for every transition step with $S' \in \alpha_{uc}(\mathsf{IS}^{\not\leftarrow}(\mathsf{AF}_i), \mathsf{IS}^{\not\leftrightarrow}(\mathsf{AF}_i), \mathsf{IS}^{\leftrightarrow}(\mathsf{AF}_i))$ we can see that $AF_i^{S'} \subset \mathsf{AF}_i$, i.e., the argumentation framework gets smaller with ever transition step. Thus, after a finite number of transitions we will eventually reach a state for which no initial set exists, e.g., the empty argumentation framework, and we have shown that every path of the transition system for α_{uc} and β_{uc} terminates, thus, the unchallenged semantics is $\alpha\beta$-closed. $\square$

Theorem 6. *Let* $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ *be an abstract argumentation framework and* $E \subseteq \mathsf{A}$. E *is an unchallenged extension if and only if either*

- $E = \emptyset$ *and* $IS^{\not\leftarrow} \cup IS^{\not\leftrightarrow}(\mathsf{AF}) = \emptyset$ *or*
- $E = E_1 \cup E_2$, $E_1 \in IS^{\not\leftarrow} \cup IS^{\not\leftrightarrow}(\mathsf{AF})$ *and* E_2 *is an unchallenged extension in* AF^{E_1}.

Proof. Let E be an unchallenged extension. If $E = \emptyset$, then per definition of the termination function β_{uc} of the unchallenged semantics we have $\mathsf{IS}^{\not\leftarrow} \cup \mathsf{IS}^{\not\leftrightarrow}(\mathsf{AF}) = \emptyset$. Assume, $E \neq \emptyset$. By definition of initial sets and the unchallenged semantics itself, there is $E_1 \in \mathsf{IS}^{\not\leftarrow}(\mathsf{AF}) \cup \mathsf{IS}^{\not\leftrightarrow}(\mathsf{AF})$ with $E_1 \subseteq E$. It remains to show that $E_2 = E \setminus E_1$ is unchallenged in AF^{E_1}. Let $a \in E_2$ and let $b_1, \ldots, b_n \in \mathsf{A}$ be the attackers of a in AF. Since E is unchallenged, there are arguments $c_1, \ldots, c_n \in E$ so that c_i attacks b_i, $i = 1, \ldots, n$ (possibly some c_i are identical). Without loss of generality, assume $c_1, \ldots, c_k \in E_1$ for some $k \leq n$. Then $b_1, \ldots, b_k$ are not present in AF^{E_1}, thus a must only be defended against $b_{k+1}, \ldots, b_n$ in AF^{E_1}. However, since $E_2 = E \setminus E_1$ we have that $c_{k+1}, \ldots, c_n \in E_2$, showing that a is defended by E_2 in AF^{E_1} and, thus, E_2 is unchallenged in AF^{E_1}.

For the other direction, if $E = \emptyset$ and $\mathsf{IS}^{\not\leftarrow} \cup \mathsf{IS}^{\not\leftrightarrow}(\mathsf{AF}) = \emptyset$ then E is also unchallenged. Assume $E = E_1 \cup E_2, E_1 \in \mathsf{IS}^{\not\leftarrow} \cup \mathsf{IS}^{\not\leftrightarrow}(\mathsf{AF})$ and E_2 is an unchallenged extension of AF^{E_1}. We have to show that E is unchallenged. Let $a \in E$ and let $b_1, \ldots, b_n \in \mathsf{A}$ be the attackers of a in AF. If $a \in E_1$ then there are $c_1, \ldots, c_n \in E_1 \subseteq E$ such that c_1 attacks b_i since E_1 is per definition either an unattacked or an unchallenged initial set of AF. If $a \in E_2$, assume for the sake of contradiction that there is an attacker b of a such that there is no $c \in E$ that attacks b in AF. It follows that b is also in AF^{E_1} and a is undefended by E_2 in AF^{E_1}. This contradicts the assumption that E_2 is unchallenged in AF^{E_1}. $\square$

Theorem 7. *The unchallenged semantics satisfies the following principles: Conflict-Freeness, Admissbility, Reduct Admissibility, Semi-Qualified Admissibility, Reinstatement, Directionality, Modularization and Serialisability.*

Proof. **Conflict-Freeness** Follows directly from Theorem 1. The unchallenged semantics is serialisable via α_{uc} and β_{uc}, thus it satisifies conflict-freeness.

Admissbility Follows directly from Theorem 1. The unchallenged semantics is serialisable via α_{uc} and β_{uc}, thus it satisifies admissibility.

Reduct Admissibility Let $\mathsf{AF} = (\mathsf{A}, \mathsf{R})$ be an argumentation framework and $E \in \mathsf{uc}(\mathsf{AF})$ is an unchallenged extension of AF. Consider some argument $a \in E$ and an argument $b \in \mathsf{A}$ that attacks a. We know that $b \in \mathsf{A} \setminus E$ because the unchallenged semantics satisfies the principle of conflict-freeness. Consider the reduct AF^E. We know that, $\mathsf{IS}^{\not\leftarrow} \cup \mathsf{IS}^{\not\leftrightarrow}(\mathsf{AF}^E) = \emptyset$ (follows from the termination function β_{uc} being true for (AF^E, E)). Therefore, $\mathsf{uc}(\mathsf{AF}^E = \{\emptyset\}$, which means $b \notin \bigcup \sigma(\mathsf{AF}^E)$. Thus, the unchallenged semantics satisfies reduct admissibility.

Semi-Qualified Admissibility Follows from the fact that admissibility is satisfied.

Reinstatement We proof by contradiction. Assume that Reinstatement does not hold for all argumentation frameworks AF for the unchallenged semantics. Then, there exists some unchallenged extension E and some argument $a \in \mathsf{A}$, such that E defends a, but we have that $a \notin E$. Consider now the corresponding state in the transition system for the extension E: (AF^E, E). Then, a must be in AF^E, since it is neither in E nor attacked by E (otherwise E would not be able to defend a as E is per definition conflict-free). Since a is defended by E and we assume that $a \notin E$, that means a must be unattacked in AF^E. If that were the case, $\{a\}$ would also be an unattacked initial set of AF^E. If such a set exists, the termination function β_{uc} would not be satisfied and α_{uc} would be able to select $\{a\}$, thus making $E \cup \{a\}$ or some superset an unchallenged extension of AF.

Weak-Reinstatement Follows from the fact that reinstatement is satisfied.

CF-Reinstatement Follows from the fact that reinstatement is satisfied.

Modularization Follows directly from Theorem 1. The unchallenged semantics is serialisable via α_{uc} and β_{uc}, thus, it satisfies modularization.

Directionality Follows from Theorem 3. The unchallenged semantics is serialisable via α_{uc} and β_{uc} and the corresponding transition system is $\alpha_{uc}\beta_{uc}$-closed as shown in Theorem 5, thus it satisfies directionality. □

Theorem 8. *The unchallenged semantics does not satisfy the following principles for all argumentation frameworks* $\mathsf{AF} \in \mathfrak{AF}$*: Strong Admissibility, Naivety, Allowing Abstention, I-Maximality and SCC-Recursiveness.*

Proof. **Naivety, I-Maximality, SCC-recursiveness** Consider the argumentation framework in Figure 3 for which neither of these three principles are satisfied.

Strong Admissibility Strong admissibility is not satisfied by the unchallenged semantics as illustrated in the following example. Consider the argumentation framework

$$\mathsf{AF} = (\{a,b,c\}, \{(a,b),(b,c),(c,a),(a,c)\}).$$

Then, $\{a\}$ is an unchallenged extension of AF, but a is not strongly defended by $\{a\}$.

Allowing Abstention Consider the argumentation framework AF_6 in Figure 5. The unchallenged extensions of AF_6 are $\{a,c\}$ and $\{f,h\}$. The argument f satisfies the premise of the allowing abstention principle, i.e., $f \in \{f,h\}$ and $f \in \{a,c\}^+$. However, there exists no $E_3 \in \mathsf{uc}(\mathsf{AF}_6)$ for which $f \notin (E_3 \cup E_3^+)$. Thus, the allowing abstention principle is not satisfied by the unchallenged semantics.

□

Theorem 9.

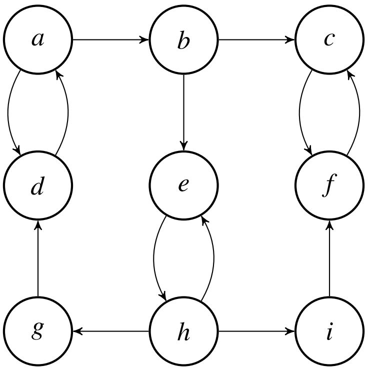

Figure 5. The argumentation framework AF_6.

1. *Ver_{uc} is in Σ_2^P and $P_{\|}^{NP}$-hard.*
2. *$Exists_{uc}^{\neg\emptyset}$ is $P_{\|}^{NP}$-complete.*
3. *$Skept_{uc}$ is Π_2^P-complete.*
4. *$Cred_{uc}$ is Σ_2^P-complete.*

For the proofs, the above theorem is split up into Propositions 2–5:

Proposition 2. *Ver_{uc} is in Σ_2^P and $P_{\|}^{NP}$-hard.*

Proof. We start by guessing an integer k. For $i = 1,\dots,k$, we iteratively guess a set $S_i \subseteq \mathsf{A}$ and verify that it is an unchallenged initial set of $\mathsf{AF}^{S_1\cup\dots\cup S_{i-1}}$. The latter can be accomplished by an NP-oracle call, since the problem is coNP-complete [6]. Finally, we verify that $\mathsf{AF}^{S_1\cup\dots\cup S_k}$ possesses no further unchallenged initial sets, which can be done in $\mathsf{P}_{\|}^{\mathsf{NP}} \subseteq \mathsf{NP}^{\mathsf{NP}}$. If $E = S_1 \cup \dots \cup S_k$, we have verified that E is an unchallenged extension.

For $\mathsf{P}_{\|}^{\mathsf{NP}}$-hardness, we reduce from the $\mathsf{P}_{\|}^{\mathsf{NP}}$-complete problem of deciding whether AF has an unchallenged initial set [6]. We assume that AF has no unattacked arguments. Then AF has an unchallenged initial set if and only if $\emptyset$ is *not* and unchallenged extension. Since $\mathsf{P}_{\|}^{\mathsf{NP}}$ is closed under complements, it follows that Ver_{uc} is $\mathsf{P}_{\|}^{\mathsf{NP}}$-hard. ☐

Proposition 3. *$Exists_{uc}^{\neg\emptyset}$ is $P_{\|}^{NP}$-complete.*

Proof. Observe that there exists a non-empty unchallenged extension if and only if AF possesses at least one unattacked or unchallenged initial set. As checking for unattacked initial sets can be ignored and the latter problem is $\mathsf{P}_{\|}^{\mathsf{NP}}$-complete [6], the claim follows. ☐

Proposition 4. *$Skept_{uc}$ is Π_2^P-complete.*

Proof. For Π_2^P-membership, we show that the complement problem $\neg Skept_{\mathsf{uc}}$, i. e, the problem of deciding that a given argument a is *not* skeptically accepted wrt. unchallenged semantics in a given AF is in $\mathsf{NP}^{\mathsf{NP}} = \Sigma_2^P = \mathrm{co}\Pi_2^P$. For that consider the following non-deterministic algorithm. We start by guessing an integer k. For $i = 1,\dots,k$, we iteratively guess a set $S_i \subseteq \mathsf{A}$ and verify that it is an unchallenged (or unattacked) initial set of $\mathsf{AF}^{S_1\cup\dots\cup S_{i-1}}$. The latter can be accomplished by an NP-oracle call, since the problem is coNP-complete [6]. Finally, we verify that $\mathsf{AF}^{S_1\cup\dots\cup S_k}$ possesses no further unchallenged

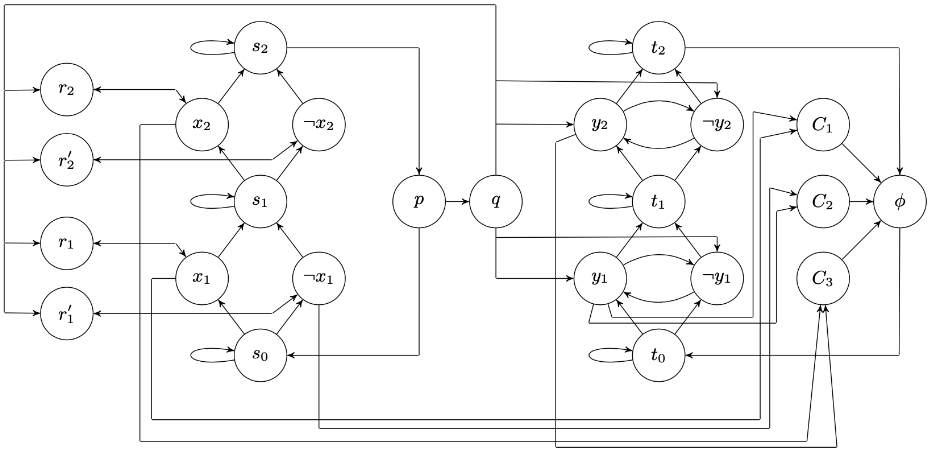

Figure 6. The argumentation framework AF_Ψ for $\Psi = \forall x_1, x_2 \exists y_1, y_2 : (x_1 \vee y_1) \wedge (\neg x_1 \vee y_1) \wedge (x_2 \vee y_2)$ from the proof of Proposition 4.

(or unattacked) initial sets, which can be done in $\mathsf{P}^{\mathsf{NP}}_{\|} \subseteq \mathsf{NP}^{\mathsf{NP}}$ [6]. It follows that $E = S_1 \cup \ldots \cup S_k$ is an unchallenged extension. If $a \notin E$, we have shown that a is not skeptically accepted wrt. unchallenged semantics in AF. This algorithm runs in Σ_2^P, so *Skept*$_{\mathsf{uc}}$ is in Π_2^P.

For Π_2^P-hardness we provide a reduction from the Π_2^P-hard problem of deciding whether a quantified Boolean formula of the form $\psi = \forall X \exists Y : \phi(X,Y)$ evaluates to true. Let $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$ and let $\phi(X,Y)$ be given as a set of clauses $\phi(X,Y) = \{C_1(X,Y), \ldots, C_o(X,Y)\}$. Construct an abstract argumentation framework $\mathsf{AF}_\Psi = (\mathsf{A}_\Psi, \mathsf{R}_\Psi)$ as follows:

$$
\begin{aligned}
\mathsf{A}_\Psi = {} & \{x_i, \neg x_i, s_i, r_i, r'_i \mid i = 1, \ldots, n\} \cup \{y_i, \neg y_i, t_i \mid i = 1 \ldots, m\} \cup \{C_1, \ldots, C_o, s_0, t_0, p, q, \phi\} \\
\mathsf{R}_\Psi = {} & \{(x_i, s_i), (\neg x_i, s_i) \mid i = 1, \ldots, n\} \cup \{(s_i, x_{i+1}), (s_i, \neg x_{i+1}) \mid i = 0, \ldots, n-1\} \\
& \cup \{(r_i, x_i), (x_i, r_i), (r'_i, \neg x_i), (\neg x_i, r'_i) \mid i = 1, \ldots, n\} \\
& \cup \{(s_n, p), (p, s_0), (p, q), (s_0, s_0), \ldots, (s_n, s_n)\} \\
& \cup \{(q, r_1), (q, r'_1), \ldots, (q, r_n), (q, r'_n)\} \\
& \cup \{(q, y_i), (q, \neg y_i) \mid i = 1, \ldots, m\} \\
& \cup \{(y_i, t_i), (\neg y_i, t_i) \mid i = 1, \ldots, m\} \cup \{(t_i, y_{i+1}), (t_i, \neg y_{i+1}) \mid i = 0, \ldots, m-1\} \\
& \cup \{(y_1, \neg y_1), (\neg y_1, y_1), \ldots, (y_m, \neg y_m), (\neg y_m, y_m)\} \\
& \cup \{(t_m, \phi), (\phi, t_0), (t_0, t_0), \ldots, (t_m, t_m)\} \\
& \cup \{(l, C_i) \mid l \in C_i, i = 1, \ldots, o\} \cup \{(C_1, \phi), \ldots, (C_o, \phi)\}
\end{aligned}
$$

Note that the construction AF_Ψ is polynomial in the size of Ψ, in particular it is $|\mathsf{A}_\Psi| = 5|X| + 3|Y| + |\phi| + 5$. Figure 6 shows an example of the construction for $\Psi = \forall x_1, x_2 \exists y_1, y_2 : (x_1 \vee y_1) \wedge (\neg x_1 \vee y_1) \wedge (x_2 \vee y_2)$. In order to understand the construction AF_Ψ, let us first consider the gadget framework $\mathsf{AF}(z_1, \ldots, z_k, u_1, \ldots, u_k, \alpha)$ depicted in

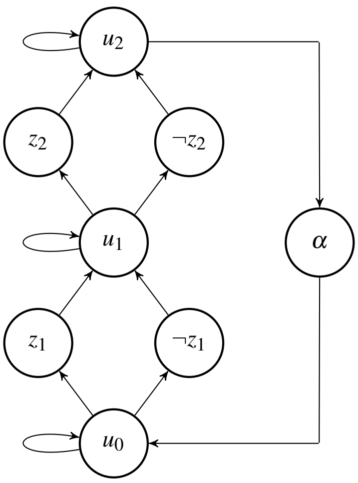

Figure 7. The gadget framework $\mathsf{AF}(z_1,\ldots,z_k,u_1,\ldots,u_k,\alpha)$ for $k=2$ from the proof of Proposition 4

Figure 7 for $k=2$ and which appears twice in AF_Ψ, namely as $\mathsf{AF}(x_1,\ldots,x_n,s_1,\ldots,s_n,p)$ and as $\mathsf{AF}(y_1,\ldots,y_n,t_1,\ldots,t_n,\phi)$. An essential feature of $\mathsf{AF}(z_1,\ldots,z_k,u_1,\ldots,u_k,\alpha)$ is that its set of initial sets is characterised via

$$\begin{aligned}\mathsf{IS}(\mathsf{AF}(z_1,\ldots,z_k,u_1,\ldots,u_k,\alpha)) &= \mathsf{IS}^{\not\leftrightarrow}(\mathsf{AF}(z_1,\ldots,z_k,u_1,\ldots,u_k,\alpha)) &(1)\\ &= \{\{\dot{z}_1,\ldots,\dot{z}_k,\alpha\} \mid \dot{z}_i \in \{z_i,\neg z_i\}, i=1,\ldots,k\} &(2)\end{aligned}$$

In other words, every maximal and consistent selection of literals from $\{z_1,\neg z_1,\ldots,z_k,\neg z_k\}$ (plus α) is an initial set (and these are the only initial sets). To see this, observe that any one of $z_i,\neg z_i$ defends both z_{i+1} and $\neg z_{i+1}$ from u_i, for $i=1,\ldots,k-1$. In addition, both of $z_n,\neg z_n$ defend α from u_k and α defends both $z_1,\neg z_1$ from u_0. As an initial set is minimal, only one argument of each pair $z_i,\neg z_i$ can be included, $i=1,\ldots,k$. Moreover, these initial sets are unchallenged, as all their attackers are self-attacking and cannot be a member of any admissible set.

Within the framework AF_Ψ, the gadget framework $\mathsf{AF}(x_1,\ldots,x_n,s_1,\ldots,s_n,p)$ plays the role of the universal quantification "$\forall X$" as it allows to start the construction of an unchallenged extension with any assignment to X. Note that the additional bidirectional attacks between the arguments $x_1,\neg x_1,\ldots,x_n,\neg x_n$ and the argument $r_1,r_1',\ldots,r_n,r_n'$, respectively, do not change statements (1) and (2), as no $r_1,r_1',\ldots,r_n,r_n'$ can be part of any initial set (if, e. g, r_1 would, it must be contained in an initial set with p, as p is the only argument defending it from q; but this implies the inclusion of x_1 or $\neg x_1$ which would be in conflict with r_1). Note also that AF_Ψ possesses no further initial sets than those from $\mathsf{AF}(x_1,\ldots,x_n,s_1,\ldots,s_n,p)$. In particular, if any y_i or $\neg y_i$, $i=1,\ldots,m$ would be a member of an initial set S, then $p\in S$ as well, as p is the only argument defending the y-arguments from q. It follows that S would strictly contain one of the initial sets from the gadget $\mathsf{AF}(x_1,\ldots,x_n,s_1,\ldots,s_n,p)$, contradicting the minimality of initial sets. For the same reason, ϕ is not a member of any initial set as its inclusion implies the inclusion of one of y_m or $\neg y_m$. Inclusion of any of the $C_1,\ldots,C_m$ would also violate minimality (and possibly conflict-freeness). Finally, all arguments $t_0,\ldots,t_m$ are self-conflicting. So we have:

$$\mathsf{IS}(\mathsf{AF}_\Psi) = \mathsf{IS}^{\not\leftarrow}(\mathsf{AF}_\Psi) = \{\{\dot{x}_1,\ldots,\dot{x}_n,p\} \mid \dot{x}_i \in \{x_i,\neg x_i\}, i = 1,\ldots,n\} \tag{3}$$

Now, observe that the arguments $p,q,r_1,r'_1,\ldots,r_n,r'_n$ and, in particular, the attacks between $r_1,r'_1,\ldots,r_n,r'_n$ and the arguments in $\{x_1,\neg x_1,\ldots,x_n,\neg x_n\}$, respectively, block the further inclusion of any of the remaining x-arguments once we move to the first reduct. More precisely, let $S \in \mathsf{IS}^{\not\leftarrow}(\mathsf{AF}_\Psi)$ be some initial set and $(\mathsf{AF}_\Psi)^S$ the resulting reduct. Observe that $q,s_0,\ldots,s_n$ and all attacks originating from these arguments are necessarily removed in $(\mathsf{AF}_\Psi)^S$ (possibly, some arguments from $C_1,\ldots,C_o$ are removed as well). It follows that the remaining r-arguments become singleton initial sets. Without loss of generality, assume $x_1 \in S$. Then $\neg x_1$ is still in $(\mathsf{AF}_\Psi)^S$ and attacks and is attacked by $\{r'_1\}$. This makes both $\{\neg x_1\}$ as well as $\{r'_1\}$ challenged initial sets, as well as all singleton sets comprised of the remaining x-arguments.

Let us now consider the arrangement involving the arguments $C_1,\ldots,C_o$ and ϕ. Note that this sub-construction is basically the same as Reduction 3.6 from [20] and is the standard construction for modelling satisfiability (with the difference that the x-arguments do not attack their complements, respectively). In essence, the formula ϕ is true (or the argument ϕ can be included in the extension under construction) if all clauses $C_1,\ldots,C_o$ are satisfied (or attacked by an accepted argument). A clause C_i, $i = 1,\ldots,o$ is satisfied, if at least one of its literals is true (or the corresponding argument is included in the extension under construction). Note that all of $\{x_1,\neg x_1,\ldots,x_n,\neg x_n,y_1,\neg y_1,\ldots,y_m,\neg y_m\}$ have corresponding attacks on the $C_1,\ldots,C_o$.

In order to complete the construction of an unchallenged extension, we now have to select an unchallenged initial set from $(\mathsf{AF}_\Psi)^S$ (note that $(\mathsf{AF}_\Psi)^S$ has no unattacked arguments). As above, the gadget $\mathsf{AF}(y_1,\ldots,y_m,t_1,\ldots,t_m,\phi)$, which is completely contained in $(\mathsf{AF}_\Psi)^S$, has the initial sets

$$\begin{aligned}\mathsf{IS}(\mathsf{AF}(y_1,\ldots,y_m,t_1,\ldots,t_m,\phi)) &= \mathsf{IS}^{\not\leftrightarrow}(\mathsf{AF}(y_1,\ldots,y_m,t_1,\ldots,t_m,\phi))\\ &= \{\{\dot{y}_1,\ldots,\dot{y}_m,\phi\} \mid \dot{y}_i \in \{y_i,\neg y_i\}, i = 1,\ldots,m\}\end{aligned}$$

However, in $(\mathsf{AF}_\Psi)^S$, the argument ϕ has to be additionally defended against the arguments $C_1,\ldots,C_o$ (at least those which are still present in $(\mathsf{AF}_\Psi)^S$). So only those sets of the form $\{\dot{y}_1,\ldots,\dot{y}_m,\phi\}$ are initial sets that attack all remaining C-arguments. In other words, ϕ can only be included in the construction of an unchallenged extension, if there is an assignment of Y that (together with the previously selected assignment of X) satisfies ϕ. To simplify the construction we assume that for the QBF $\psi = \forall X \exists Y : \phi(X,Y)$ and any assignment for X, there is also only at most one assignment for Y, such that $\phi(X,Y)$ is true. This ensures that the initial set is unchallenged.

With the above preliminary thoughts in mind, we show now that Ψ evaluates to true if and only if ϕ is skeptically accepted wrt. unchallenged semantics in AF_Ψ.

- "$\Rightarrow$": Let Ψ evaluate to true and assume that ϕ is *not* skeptically accepted wrt. unchallenged semantics in AF_Ψ. Then there is an unchallenged extension E with $\phi \notin E$. Due to (3), $S = \{\dot{x}_1,\ldots,\dot{x}_n,p\} \subseteq E$ for some $\dot{x}_i \in \{x_i,\neg x_i\}, i = 1,\ldots,n$. Since $\phi \notin E$ and $(\mathsf{AF}_\Psi)^S$ can only contain initial sets of the form $\{\dot{y}_1,\ldots,\dot{y}_m,\phi\}$, it follows that $E = S$ and $(\mathsf{AF}_\Psi)^S$ has no further unchallenged (or unattacked) initial sets. In other words, given the assignment S of the X-variables there is no assignment of the Y-variables such that $\phi(X,Y)$ is true. It follows that Ψ evaluates to false, contradicting the assumption.

- “$\Leftarrow$”: Let ϕ be skeptically accepted wrt. unchallenged semantics in AF_Ψ. Recall that all possible combinations of the X-variables have a 1:1 correspondence with initial sets of AF_Ψ, see Equation (3), so for every assignment of the X-variables there is a corresponding unchallenged extension. Since all these extensions include ϕ, there must be a selection of the y-arguments that (together with the selection of the x-arguments) defending ϕ. In other words, Ψ evaluates to true.

It follows that $\mathit{Skept}_{\mathsf{uc}}$ is Π_2^P-hard. ☐

Proposition 5. *Cred_{uc} is Σ_2^P-complete.*

Proof. For Σ_2^P-membership, we use the same algorithm as in the proof of Proposition 4. More precisely, we start by guessing an integer k. For $i = 1, \ldots, k$, we iteratively guess a set $S_i \subseteq \mathsf{A}$ and verify that it is an unchallenged (or unattacked) initial set of $\mathsf{AF}^{S_1 \cup \ldots \cup S_{i-1}}$. The latter can be accomplished by an NP-oracle call, since the problem is coNP-complete [6]. Finally, we verify that $\mathsf{AF}^{S_1 \cup \ldots \cup S_k}$ possesses no further unchallenged (or unattacked) initial sets, which can be done in $\mathsf{P}_{\|}^{\mathsf{NP}} \subseteq \mathsf{NP}^{\mathsf{NP}}$ [6]. It follows that $E = S_1 \cup \ldots \cup S_k$ is an unchallenged extension. For $a \in E$, we have shown that a is credulously accepted wrt. unchallenged semantics in AF. This algorithm runs in Σ_2^P, so $\mathit{Cred}_{\mathsf{uc}}$ is in Σ_2^P.

For Σ_2^P-hardness we provide a reduction from the Σ_2^P-hard problem of deciding whether a quantified Boolean formula of the form $\psi = \forall X \exists Y : \phi(X,Y)$ evaluates to false. For that we slightly extend the construction $\mathsf{AF}_\Psi = (\mathsf{A}_\Psi, \mathsf{R}_\Psi)$ from the proof of Proposition 4 as follows. Let $\mathsf{AF}'_\Psi = (\mathsf{A}'_\Psi, \mathsf{R}'_\Psi)$ be defined via

$$\mathsf{A}'_\Psi = \mathsf{A}_\Psi \cup \{\tilde{\phi}\}$$

$$\mathsf{R}'_\Psi = \mathsf{R}_\Psi \cup \{(\tilde{\phi}, \phi), (\phi, \tilde{\phi}), (q, \tilde{\phi})\}$$

Compared to AF_Ψ we add another argument $\tilde{\psi}$ with a bidirectional attack to ϕ and an attack from q to $\tilde{\phi}$. The latter attack's function is to prohibit $\{\tilde{\phi}\}$ being an initial set of AF'_Ψ. Observe that $\mathsf{IS}(\mathsf{AF}'_\Psi) = \mathsf{IS}(\mathsf{AF}_\Psi)$. Given $S \in \mathsf{IS}(\mathsf{AF}'_\Psi)$, it follows that $\{\tilde{\phi}\}$ is an unchallenged initial set of $(\mathsf{AF}'_\Psi)^S$ if and only if there is no initial set containing ϕ, which—following the argumentation in the proof of Proposition 4—only happens when there is a unique combination of the Y-variables such that ϕ evaluates to true. So ϕ is credulously accepted wrt. unchallenged semantics if and only if $\forall X \exists Y : \phi(X,Y)$ evaluates to false. ☐