Measuring Disagreement with Interpolants

Proofs of technical results

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A Proofs for Section 3

Lemma 1. *Let ϕ be a propositional formula and x one of the propositional variables occurring in ϕ. Then the following hold:*

(1) $\phi \models \exists x\phi$;
(2) Let ψ be a formula with $\mathsf{At}(\psi) \subseteq \mathsf{At}\backslash\{x\}$ such that $\phi \models \psi$. Then $\exists x\phi \models \psi$.
(3) $\forall x\phi \models \phi$;
(4) Let ψ be a formula with $\mathsf{At}(\psi) \subseteq \mathsf{At}\backslash\{x\}$ such that $\psi \models \phi$. Then $\psi \models \forall x\phi$.

Proof. (1) Let $\omega : \mathsf{At} \to \{\mathsf{true}, \mathsf{false}\}$ be such that $\omega \models \phi$. If $\omega(x) = \mathsf{false}$ then $\omega \models \phi[x \mapsto \perp]$; if $\omega(x) = \top$ then $\omega \models \phi[x \mapsto \top]$. Thus, $\omega \models \exists x\phi$.

(2) Let ψ be a formula with $\mathsf{At}(\psi) \subseteq \mathsf{At}\backslash\{x\}$ such that $\phi \models \psi$. Let $\omega : \mathsf{At} \to \{\mathsf{true}, \mathsf{false}\}$ be such that $\omega \models \exists x\phi$. Then $\omega \models \phi[x \mapsto \perp] \vee \phi[x \mapsto \top]$, so $\omega \models \phi[x \mapsto \perp]$ or $\omega \models \phi[x \mapsto \top]$. Assume that $\omega \models \phi[x \mapsto \perp]$ (the other case is analogous). Let $\omega' : \mathsf{At} \to \{\mathsf{true}, \mathsf{false}\}$ defined by $\omega'(x) = \mathsf{false}$ and $\omega'(y) = \omega(y)$ for all $y \neq x$. Then $\omega' \models \phi$, so as $\phi \models \psi$ we have $\omega' \models \psi$. But since x does not occur in ψ and ω and ω' agree for all atoms but x, we have $\omega \models \psi$.

(3) Let $\omega : \mathsf{At} \to \{\mathsf{true}, \mathsf{false}\}$ be such that $\omega \models \forall x\phi$. Then $\omega \models \phi[x \mapsto \perp] \wedge \phi[x \mapsto \top]$. Assume $\omega \not\models \phi$. If $\omega(x) = \mathsf{false}$ it follows that $\omega \not\models \phi[x \mapsto \perp]$; if $\omega(x) = \mathsf{true}$ it follows that $\omega \not\models \phi[x \mapsto \top]$. Contradiction.

(4) Let ψ be a formula with $\mathsf{At}(\psi) \subseteq \mathsf{At}\backslash\{x\}$ such that $\psi \models \phi$. Let $\omega : \mathsf{At} \to \{\mathsf{true}, \mathsf{false}\}$ be such that $\omega \models \psi$. Since $\psi \models \phi$ it follow that $\omega \models \phi$. Clearly, $\omega(x) = \mathsf{false}$ or $\omega(x) = \mathsf{true}$. Assume $\omega(x) = \mathsf{false}$ (the other case is analogous). Then $\omega \models \phi[x \mapsto \perp]$. Let $\omega' : \mathsf{At} \to \{\mathsf{false}, \mathsf{true}\}$ defined by $\omega'(x) = \mathsf{true}$ and $\omega'(y) = \omega(y)$ for all $y \neq x$. Since x does not occur in ψ and ω and ω' agree for all atoms but x, we have $\omega' \models \psi$, so $\omega' \models \phi$, i.e. $\omega' \models \phi[x \mapsto \top]$, hence $\omega \models \phi[x \mapsto \top]$ (as $\phi[x \mapsto \top]$ does not contain any occurrences of x).

We thus showed that $\omega \models \phi[x \mapsto \perp] \wedge \phi[x \mapsto \top]$, i.e. $\omega \models \forall x\phi$.

Proposition 1. Let Φ be a finite set of propositional formulas. Assume that $\mathsf{At}(\Phi) = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$. Then the following hold:

(1) $\Phi \models \exists x_1 \ldots \exists x_n \Phi$.
(2) Let ψ be a propositional formula with $\mathsf{At}(\psi) \subseteq \{y_1, \ldots, y_m\}$ such that $\Phi \models \psi$. Then $\exists x_1 \ldots \exists x_n \Phi \models \psi$.

(3) $\forall x_1 \dots \forall x_n \Phi \models \Phi$.
(4) Let ψ be a formula with $\mathsf{At}(\Psi) \subseteq \{y_1, \dots, y_m\}$ such that $\psi \models \Phi$ (i.e. $\psi \models \phi_i$ for every formula $\phi_i \in \Phi$). Then $\psi \models \forall x_1 \dots \forall x_n \Phi$.

Proof. Assume that $\Phi = \{\phi_1, \dots, \phi_n\}$, and let $\phi = \phi_1 \wedge \dots \wedge \phi_n$. All results follow from Lemma 1 (applied to ϕ) by induction on the number of variables which are eliminated.

In what follows, if Φ is a finite set of formulae, say $\Phi = \{\phi_1, \dots, \phi_n\}$ we denote by $\neg(\wedge\Phi)$ the formula $\neg(\phi_1 \wedge \dots \wedge \phi_n)$.

Lemma 2. *Let Φ and Φ' be finite sets of propositional formulas with $\mathsf{At}(\Phi) = \{x_1, \dots, x_n, y_1, \dots, y_m\}$ and $\mathsf{At}(\Phi') = \{y_1, \dots, y_m, z_1, \dots, z_k\}$ and such that $\Phi \cup \Phi'$ is unsatisfiable (i.e. $\Phi \models \neg(\wedge\Phi')$).*
Let $\phi_w := \forall z_1 \dots \forall z_k \neg(\wedge\Phi')(y_1, \dots, y_m, z_1, \dots, z_k)$. Then the following hold:

(i) $\Phi(x_1, \dots, x_n, y_1, \dots, y_n) \models \phi_w(y_1, \dots, y_m)$;
(ii) $\phi_w(y_1, \dots, y_m) \wedge \Phi'(y_1, \dots, y_m, z_1, \dots, z_k) \models \perp$;
(iii) *ϕ_w contains only propositional variables occurring in both Φ and Φ';*
(iv) *For every interpolant ϕ of Φ and Φ', $\phi \models \phi_w$.*

Proof. (i) can be proved by induction on n using Lemma 1(4), using the fact that $\Phi \models \neg(\wedge\Phi')$; (ii) can be proved by induction on n using Lemma 1(3). (iii) is obvious. To prove (iv) let ϕ be an interpolant of Φ and Φ'. Then, in particular, $\Phi' \cup \{\phi\} \models \perp$, so $\phi \models \neg(\wedge\Phi')$. By Lemma 1(4) it follows that $\phi \models \forall z_1 \dots \forall z_k \neg(\wedge\Phi')(y_1, \dots, y_m, z_1, \dots, z_k)$, i.e. $\phi \models \phi_w$.

Proposition 3. Let Φ and Φ' be finite set of formulas.

1. If $[\phi], [\phi'] \in \mathbb{SI}(\Phi, \Phi')$ then $[\phi \wedge \phi'] \in \mathbb{SI}(\Phi, \Phi')$.
2. If $[\phi], [\phi'] \in \mathbb{SI}(\Phi, \Phi')$ then $[\phi \vee \phi'] \in \mathbb{SI}(\Phi, \Phi')$.
3. There is a uniquely defined $[\phi_w] \in \mathbb{SI}(\Phi, \Phi')$ with $\phi' \models \phi_w$ for all $[\phi'] \in \mathbb{SI}(\Phi, \Phi')$.
4. There is a uniquely defined $[\phi_s] \in \mathbb{SI}(\Phi, \Phi')$ with $\phi_s \models \phi'$ for all $[\phi'] \in \mathbb{SI}(\Phi, \Phi')$.

Proof. (1) If $[\phi], [\phi'] \in \mathbb{SI}(\Phi, \Phi')$ then $\phi, \phi' \in \mathbb{I}(\Phi, \Phi')$ so: (i) $\Phi \models \phi$ and $\Phi \models \phi'$, so $\Phi \models \phi \wedge \phi'$; (ii) $\Phi' \cup \{\phi\} \models \perp$, so $\Phi' \cup \{\phi \wedge \phi'\} \models \perp$ and (iii) $\mathsf{At}(\phi), \mathsf{At}(\phi') \subseteq \mathsf{At}(\Phi) \cap \mathsf{At}(\Phi')$, hence this is the case also for $\phi \wedge \phi'$. Thus, $\phi \wedge \phi' \in \mathbb{I}(\Phi, \Phi')$, so $[\phi \wedge \phi'] \in \mathbb{SI}(\Phi, \Phi')$.

(2) If $[\phi], [\phi'] \in \mathbb{SI}(\Phi, \Phi')$ then $\phi, \phi' \in \mathbb{I}(\Phi, \Phi')$, so (i) $\Phi \models \phi$ so $\Phi \models \phi \vee \phi'$; (ii) $\Phi' \cup \{\phi\} \models \perp$ and $\Phi' \cup \{\phi'\} \models \perp$ hence $\Phi' \cup \{\phi \vee \phi'\} \models \perp$ and (iii) $\mathsf{At}(\phi), \mathsf{At}(\phi') \subseteq \mathsf{At}(\Phi) \cap \mathsf{At}(\Phi')$, hence this is the case also for $\phi \vee \phi'$. Thus, $\phi \vee \phi' \in \mathbb{I}(\Phi, \Phi')$, so $[\phi \vee \phi'] \in \mathbb{SI}(\Phi, \Phi')$.

For proving (3) and (4) assume that $\mathsf{At}(\Phi) = \{x_1, \dots, x_n, y_1, \dots, y_m\}$ and $\mathsf{At}(\Phi') = \{y_1, \dots, y_m, z_1, \dots, z_k\}$.

(3) Let $\phi_w := \forall z_1 \dots \forall z_k \neg(\wedge\Phi')(y_1, \dots, y_m, z_1, \dots, z_k)$. Then, by Lemma 2, the following hold:

- $\Phi(x_1,\ldots,x_n,y_1,\ldots,y_n) \models \phi_w(y_1,\ldots,y_m)$;
- $\phi_w(y_1,\ldots,y_m) \wedge \Phi'(y_1,\ldots,y_m,z_1,\ldots,z_k) \models\perp$;
- ϕ_w contains only propositional variables occurring in both Φ and Φ';
- For every interpolant ϕ of Φ and Φ', $\phi \models \phi_w$.

Clearly, $[\phi_w]$ is uniquely determined. If $[\phi]$ is a different class with the same property, then $\phi' \models \phi$ for all $[\phi'] \in \mathbb{SI}(\Phi,\Phi')$, hence also $\phi_w \models \phi$. On the other hand, if $\phi \in \mathbb{SI}(\Phi,\Phi')$ then, by Lemma 2(iv), $\phi \models \phi_w$. Thus, $[\phi_w] = [\phi]$.

(4) Let $\phi_s := \exists x_1 \ldots \exists x_k \Phi(x_1,\ldots,x_n,y_1,\ldots,y_n)$. Then, by Proposition 1, the following hold:

- $\Phi(x_1,\ldots,x_n,y_1,\ldots,y_n) \models \phi_s(y_1,\ldots,y_m)$;
- $\phi_s(y_1,\ldots,y_m) \wedge \Phi'(y_1,\ldots,y_m,z_1,\ldots,z_k) \models\perp$;
- ϕ_s contains only propositional variables occurring in both Φ and Φ';
- For every interpolant ϕ of Φ and Φ', $\phi_s \models \phi$.

To show that $[\phi_s]$ is uniquely determined one can use a proof analogous to the one used for $[\phi_w]$.

B Proofs for Section 5

Theorem 2. *The compliance of the measures $\mathcal{D}^{\Sigma}_{\mathbb{SI}}$, $\mathcal{D}^{\max}_{\mathbb{SI}}$, $\mathcal{D}^{\Sigma}_{J}$, and $\mathcal{D}^{\max}_{J}$ is as shown in Table 1.*

	MO	DO	SFI	AI	TA	CO	MAJ	MAJL
$\mathcal{D}^{\Sigma}_{\mathbb{SI}}$	✓	✗	✓	✓	✗	✓	✗	✗
$\mathcal{D}^{\max}_{\mathbb{SI}}$	✓	✗	✓	✓	✓	✓	✗	✗
$\mathcal{D}^{\Sigma}_{J}$	✓	✗	✓	✓	✓	✓	✗	✗
$\mathcal{D}^{\max}_{J}$	✓	✗	✓	✓	✓	✓	✗	✗

Table 1. Compliance of investigated disagreement measures with postulates.

Proof. First, observe that when properties MO, DO, SFI, AI, TA and CO are satisfied for a function $\mathcal{D} : \mathbb{K} \times \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ then they are also satisfied for $\mathcal{D}^{\Sigma}$ and $\mathcal{D}^{\max}$. Below, we only show the former and therefore only consider $\mathcal{D}_{\mathbb{SI}}$ and $\mathcal{D}_J$.

- $\mathcal{D}_{\mathbb{SI}}$
  MO Follows from Proposition 4, item 3 and 4.
  DO Consider the following counter-example. Let $\mathcal{K}_1 = \{\top\}$ and $\mathcal{K}_2 = \{\neg p, q\}$. Let ϕ stand for the formula p, and ψ stand for $(p \vee q) \wedge (p \vee \neg q)$. Note that $\mathbb{SI}(\phi,\mathcal{K}_2) = \{[p]\}$ and $\mathbb{SI}(\psi,\mathcal{K}_2) = \{[p],[p \vee \neg q]\}$, and $\mathbb{SI}(\mathcal{K}_2,\phi) = \{[\neg p]\}$ and $\mathbb{SI}(\mathcal{K}_2,\psi) = \{[\neg p],[\neg p \wedge q]\}$. Thus, $\mathcal{D}^{max}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\phi\},\mathcal{K}_2) = 1$ and $\mathcal{D}^{max}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\psi\},\mathcal{K}_2) = 2$, while $\mathcal{D}^{\Sigma}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\phi\},\mathcal{K}_2) = 2$ and $\mathcal{D}^{\Sigma}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\psi\},\mathcal{K}_2) = 4$. Note that $\phi \models \psi$ and $\phi \not\models \perp$, but $\mathcal{D}^{\max}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\phi\},\mathcal{K}_2) \not\geq \mathcal{D}^{max}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\psi\},\mathcal{K}_2)$ and $\mathcal{D}^{\Sigma}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\phi\},\mathcal{K}_2) \not\geq \mathcal{D}^{\Sigma}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\psi\},\mathcal{K}_2)$.

SFI Let $P = \langle \mathcal{K}_1, \cdots, \mathcal{K}_n \rangle$ be a knowledge profile, and ϕ a consistent formula such that $At(\phi) \cap \big(\cup_{\mathcal{K}_i \in P} At(\mathcal{K}_i) \big) = \emptyset$. This implies that, for each $\mathcal{K}_i \in P$, $At(\mathcal{K}_i) \cap At(\phi) = \emptyset$. To show that $\mathcal{D}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\phi\}, \cdots, \mathcal{K}_n) = \mathcal{D}_{\mathbb{SI}}(\mathcal{K}_1, \cdots, \mathcal{K}_n)$, it suffice to show that for each $\mathcal{K}_i \in P$, $\mathbb{SI}(\mathcal{K}_1, \mathcal{K}_i) = \mathbb{SI}(\mathcal{K}_1 \cup \{\phi\}, \mathcal{K}_i)$ and $\mathbb{SI}(\mathcal{K}_i, \mathcal{K}_1) = \mathbb{SI}(\mathcal{K}_i, \mathcal{K}_1 \cup \{\phi\})$. The case that $\mathcal{K}_1$ is inconsistent is trivial. So we focus only on the case that $\mathcal{K}_1$ is consistent. From Proposition 4, items 3 and 4, we already have that $\mathbb{SI}(\mathcal{K}_1, \mathcal{K}_i) \subseteq \mathbb{SI}(\mathcal{K}_1 \cup \{\phi\}, \mathcal{K}_i)$ and $\mathbb{SI}(\mathcal{K}_i, \mathcal{K}_1) \subseteq \mathbb{SI}(\mathcal{K}_i, \mathcal{K}_1 \cup \{\phi\})$. So we only need to show that $\mathbb{SI}(\mathcal{K}_1 \cup \{\phi\}, \mathcal{K}_i) \subseteq \mathbb{SI}(\mathcal{K}_1, \mathcal{K}_i)$ and $\mathbb{SI}(\mathcal{K}_i, \mathcal{K}_1 \cup \{\phi\}) \subseteq \mathbb{SI}(\mathcal{K}_i, \mathcal{K}_1)$. Let $[\psi] \in \mathbb{SI}(\mathcal{K}_1 \cup \{\phi\}, \mathcal{K}_i)$, we need to show that $[\psi] \in \mathbb{SI}(\mathcal{K}_1, \mathcal{K}_i)$. Thus, for some formula $\alpha \equiv \psi$, we have $\mathcal{K}_1 \cup \{\phi\} \models \alpha$, $\mathcal{K}_i \cup \{\alpha\} \models \bot$ and $At(\alpha) \subseteq At(\mathcal{K}_1 \cup \{\phi\}) \cap At(\mathcal{K}_i)$ which implies that $At(\alpha) \subseteq At(\mathcal{K}_1 \cup \{\phi\})$. Thus, as $At(\phi) \cap At(\mathcal{K}_1) = \emptyset$, we get $At(\alpha) \subseteq At(\mathcal{K}_1)$. In summary, we have ϕ is consistent, $At(\mathcal{K}_1) \cap At(\phi) = \emptyset$, $At(\alpha) \subseteq At(\mathcal{K}_1)$ and $\mathcal{K}_1 \cup \{\phi\} \models \alpha$. Therefore, from Proposition 6, we get $\mathcal{K}_1 \models \alpha$. Therefore, $[\alpha] \in \mathbb{SI}(\mathcal{K}_1, \mathcal{K}_i)$. Thus, as $\alpha \equiv \psi$, we have that $[\psi] \in \mathbb{SI}(\mathcal{K}_1, \mathcal{K}_i)$. The proof for $\mathbb{SI}(\mathcal{K}_i, \mathcal{K}_1 \cup \{\phi\}) \subseteq \mathbb{SI}(\mathcal{K}_i, \mathcal{K}_1)$ is analogous.

AI Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two knowledge bases, and ϕ and ψ be two formulas. First, note that $At(\mathcal{K}_1 \cup \{\phi, \psi\}) = At(\mathcal{K}_1 \cup \{\phi \wedge \psi\})$, and for every formula α, $\mathcal{K}_1 \cup \{\phi, \psi\} \models \alpha$ iff $\mathcal{K}_1 \cup \{\phi \wedge \psi\} \models \alpha$. Therefore, $\mathbb{SI}(\mathcal{K}_1 \cup \{\phi, \psi\}, \mathcal{K}_2) = \mathbb{SI}(\mathcal{K}_1 \cup \{\phi \wedge \psi\}, \mathcal{K}_2)$ and $\mathbb{SI}(\mathcal{K}_2, \mathcal{K}_1 \cup \{\phi, \psi\}) = \mathbb{SI}(\mathcal{K}_2, \mathcal{K}_1 \cup \{\phi \wedge \psi\})$. Thus, $\mathcal{D}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\varphi, \psi\}, \mathcal{K}_2) = \mathcal{D}_{\mathbb{SI}}(\mathcal{K}_1 \cup \{\phi \wedge \psi\}, \mathcal{K}_2)$.

TA We will show that **TA** is not satisfied by $\mathcal{D}^{\Sigma}_{\mathbb{SI}}$, but it is satisfied by $\mathcal{D}^{\max}_{\mathbb{SI}}$. For $\mathcal{D}^{\Sigma}_{\mathbb{SI}}$, consider the following counter-example. Let $\mathcal{K}_1 = \{p \wedge \neg p\}, \mathcal{K}_2 = \{q\}$ and $\mathcal{K}_\top = \{p \vee \neg p\}$. Note that $\top \models \mathcal{K}_\top$, we will show that $\mathcal{D}^{\Sigma}_{\mathbb{SI}}(\mathcal{K}_1, \mathcal{K}_2) < \mathcal{D}^{\Sigma}_{\mathbb{SI}}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_\top)$. Note that $At(\mathcal{K}_1) \cap At(\mathcal{K}_2) = \emptyset$ which means that $\mathbb{SI}(\mathcal{K}_1, \mathcal{K}_2) = \{[\bot]\}$ and $\mathbb{SI}(\mathcal{K}_2, \mathcal{K}_1) = \{[\top]\}$. Thus, $\mathcal{D}^{\Sigma}_{\mathbb{SI}}(\mathcal{K}_1, \mathcal{K}_2) = 2$. Note that $\mathbb{SI}(\mathcal{K}_1, \mathcal{K}_\top) = \{[\bot]\}$ which implies that $\mathcal{D}_{\mathbb{SI}}(\mathcal{K}_1, \mathcal{K}_\top) = 1$. Therefore, $3 \leq \mathcal{D}_{\mathbb{SI}}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_\top)$. Thus, as $\mathcal{D}^{\Sigma}_{\mathbb{SI}}(\mathcal{K}_1, \mathcal{K}_2) = 2$, we get that $\mathcal{D}^{\Sigma}_{\mathbb{SI}}(\mathcal{K}_1, \mathcal{K}_2) < \mathcal{D}^{\Sigma}_{\mathbb{SI}}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_\top)$. For $\mathcal{D}^{\max}_{\mathbb{SI}}$, let us suppose for contradiction that it does not satisfy **TA**. Thus, for some $\mathcal{K}_\top$ and some knowledge base profile $P = \langle \mathcal{K}_1, \cdots, \mathcal{K}_n \rangle$: $\mathcal{D}^{\max}_{\mathbb{SI}}(P) \not\geq \mathcal{D}^{\max}_{\mathbb{SI}}(P \circ \mathcal{K}_\top)$, and $\top \models \mathcal{K}_\top$. This means that $\mathcal{D}^{\max}_{\mathbb{SI}}(P \circ \mathcal{K}_\top) > \mathcal{D}^{\max}_{\mathbb{SI}}(P)$. Let $S = \mathcal{D}^{\max}_{\mathbb{SI}}(P)$, then for all $\mathcal{K}_i, \mathcal{K}_j \in P, |\mathbb{SI}(\mathcal{K}_i, \mathcal{K}_j)| \leq S$. Moreover, as $\mathcal{D}^{\max}_{\mathbb{SI}}(P \circ \mathcal{K}_\top) > \mathcal{D}^{max}_{\mathbb{SI}}(P)$, we have that for some $m \in \{1, \cdots, n\}$, either $|\mathbb{SI}(\mathcal{K}_m, \mathcal{K}_\top)| > S$ or $|\mathbb{SI}(\mathcal{K}_\top, \mathcal{K}_m)| > S$. First note that both $\mathbb{SI}(\mathcal{K}_m, \mathcal{K}_\top)$ and $\mathbb{SI}(\mathcal{K}_\top, \mathcal{K}_m)$ have at most one interpolant, which is either $[\bot]$ or $[\top]$. This means that $\mathbb{SI}(\mathcal{K}_\top, \mathcal{K}_m)$ and $\mathbb{SI}(\mathcal{K}_m, \mathcal{K}_\top)$ have sizes 0 or 1. As either $|\mathbb{SI}(\mathcal{K}_\top, \mathcal{K}_m)|$ or $|\mathbb{SI}(\mathcal{K}_m, \mathcal{K}_\top)|$ is strictly greater than S, we get that $S < 1$, and either $|\mathbb{SI}(\mathcal{K}_\top, \mathcal{K}_m)| = 1$ or $|\mathbb{SI}(\mathcal{K}_m, \mathcal{K}_\top)| = 1$. In either case, $\mathcal{K}_m \models \bot$. But as $S < 1$, we get that $S = 0$ which implies that for all $\mathcal{K}_i \in P, \mathcal{K}_i \not\models \bot$ which contradicts the hypothesis that $\mathcal{K}_m \models \bot$. Therefore, $\mathcal{D}^{\max}_{\mathbb{SI}}$ satisfies **TA**.

– $\mathcal{D}_J$

MO If $\mathcal{K}_1 \cup \mathcal{K}_2$ is consistent then $\mathcal{D}_J(\mathcal{K}_1, \mathcal{K}_2) = 0$ and the property is satisfied. If $J(\mathsf{Weakest}(\mathcal{K}_1, \mathcal{K}_2)) = 0$ then $J(\mathsf{Weakest}(\mathcal{K}_1 \cup \mathcal{K}', \mathcal{K}_2)) = 0$ as well due to Proposition 4, item 6, and Definition 8, item 3. In that case $\mathcal{D}_J(\mathcal{K}_1, \mathcal{K}_2) = \mathcal{D}_J(\mathcal{K}_1 \cup \mathcal{K}', \mathcal{K}_2) = \infty$. If $J(\mathsf{Weakest}(\mathcal{K}_1, \mathcal{K}_2)) \neq 0$ and $J(\mathsf{Weakest}(\mathcal{K}_1 \cup \mathcal{K}', \mathcal{K}_2)) = 0$ then $\mathcal{D}_J(\mathcal{K}_1, \mathcal{K}_2) < \mathcal{D}_J(\mathcal{K}_1 \cup \mathcal{K}', \mathcal{K}_2) = \infty$. It remains the case that $\mathcal{K}_1 \cup \mathcal{K}_2$ is inconsistent (and so is $\mathcal{K}_1 \cup \mathcal{K}' \cup \mathcal{K}_2$) and $J(\mathsf{Weakest}(\mathcal{K}_1, \mathcal{K}_2)) \neq 0$ and $J(\mathsf{Weakest}(\mathcal{K}_1 \cup \mathcal{K}', \mathcal{K}_2)) \neq 0$. Then observe

$$\begin{aligned}\mathcal{D}_J(\mathcal{K}_1 \cup \mathcal{K}', \mathcal{K}_2) &= \frac{J(\mathsf{Strongest}(\mathcal{K}_1 \cup \mathcal{K}', \mathcal{K}_2))}{J(\mathsf{Weakest}(\mathcal{K}_1 \cup \mathcal{K}', \mathcal{K}_2))} \\ &\geq \frac{J(\mathsf{Strongest}(\mathcal{K}_1, \mathcal{K}_2))}{J(\mathsf{Weakest}(\mathcal{K}_1, \mathcal{K}_2))} = \mathcal{D}_J(\mathcal{K}_1, \mathcal{K}_2)\end{aligned}$$

Note that the inequality is due to Proposition 4, items 6 and 8, and Definition 8, item 3.

DO We recall the counter-example we used on **DO** for $\mathcal{D}_{\mathbb{SI}}$. Let $\mathcal{K}_1 = \{\top\}$ and $\mathcal{K}_2 = \{\neg p, q\}$. Let ϕ stand for the formula p, and ψ stand for $(p \vee q) \wedge (p \vee \neg q)$, and J_M be the information measure from the example of Definition 8. We summarize below the interpolants, the weakest and and strongest interpolants of ϕ and ψ modulo $\mathcal{K}_2$. We also indicate the value of $\mathcal{D}_{J_M}$ for ϕ and ψ modulo $\mathcal{K}_2$.

	$(\phi, \mathcal{K}_2)$	$(\mathcal{K}_2, \phi)$	$(\psi, \mathcal{K}_2)$	$(\mathcal{K}_2, \psi)$
$\mathbb{SI}$	$\{[p]\}$	$\{[\neg p]\}$	$\{[p], [p \vee \neg q]\}$	$\{[\neg p], [\neg p \wedge q]\}$
Weakest	$[p]$	$[\neg p]$	$[p \vee \neg q]$	$[\neg p]$
Strongest	$[p]$	$[\neg p]$	$[p]$	$[\neg p \wedge q]$
$\mathcal{D}_{J_M}$	1	1	3/2	2

Thus, $\mathcal{D}^{\Sigma}_{J_M}(\mathcal{K}_1 \cup \{\phi\}, \mathcal{K}_2) = 2$ and $\mathcal{D}^{\Sigma}_{J_M}(\mathcal{K}_1 \cup \{\psi\}, \mathcal{K}_2) = 7/2$, while $\mathcal{D}^{\max}_{J_M}(\mathcal{K}_1 \cup \{\phi\}, \mathcal{K}_2) = 1$ and $\mathcal{D}^{\max}_{J_M}(\mathcal{K}_1 \cup \{\phi\}, \mathcal{K}_2) = 2$. Note that $\phi \models \psi$ and $\varphi \not\models \bot$, but $\mathcal{D}^{\Sigma}_{J_M}(\mathcal{K}_1 \cup \{\phi\}, \mathcal{K}_2) \not\geq \mathcal{D}^{\Sigma}_{J_M}(\mathcal{K}_1 \cup \{\psi\}, \mathcal{K}_2)$ and $\mathcal{D}^{\max}_{J_M}(\mathcal{K}_1 \cup \{\phi\}, \mathcal{K}_2) \not\geq \mathcal{D}^{\max}_{J_M}(\mathcal{K}_1 \cup \{\psi\}, \mathcal{K}_2)$.

SFI It suffices to show that $\mathbb{SI}(\mathcal{K}_1, \mathcal{K}_i) = \mathbb{SI}(\mathcal{K}_1 \cup \{\phi\}, \mathcal{K}_i)$ and $\mathbb{SI}(\mathcal{K}_i, \mathcal{K}_1) = \mathbb{SI}(\mathcal{K}_i, \mathcal{K}_1 \cup \{\phi\})$. We have already shown in the proof of $\mathcal{D}_{\mathbb{SI}}$ for **SFI**, that this is indeed the case. Therefore, $\mathcal{D}_J$ satisfies **SFI**.

AI Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two finite knowledge bases. Fist, note that $\mathbb{SI}(\mathcal{K}_1 \cup \{\phi, \psi\}, \mathcal{K}_2) = \mathbb{SI}(\mathcal{K}_1 \cup \{\phi \wedge \psi\}, \mathcal{K}_2)$ and $\mathbb{SI}(\mathcal{K}_2, \mathcal{K}_1 \cup \{\phi, \psi\}) = \mathbb{SI}(\mathcal{K}_2, \mathcal{K}_1 \cup \{\phi \wedge \psi\})$. This means that $\mathsf{Strongest}(\mathcal{K}_1 \cup \{\phi, \psi\}, \mathcal{K}_2) = \mathsf{Strongest}(\mathcal{K}_1 \cup \{\phi \wedge \psi\}, \mathcal{K}_2)$ and $\mathsf{Weakest}(\mathcal{K}_1 \cup \{\phi, \psi\}, \mathcal{K}_2) = \mathsf{Weakest}(\mathcal{K}_1 \cup \{\phi \wedge \psi\}, \mathcal{K}_2)$. Thus, $\mathcal{D}_J(\mathcal{K}_1 \cup \{\phi, \psi\}, \mathcal{K}_2) = \mathcal{D}_J(\mathcal{K}_1 \cup \{\phi \wedge \psi\}, \mathcal{K}_2)$.

TA Let $P = \langle \mathcal{K}_1, \cdots, \mathcal{K}_n \rangle$ be a knowledge profile, and $\mathcal{K}_\top$ be a knowledge base such that $\top \models \mathcal{K}_\top$. If $\mathcal{D}^{\Sigma}_J(P) = \infty$ and $\mathcal{D}^{\max}_J(P) = \infty$ then $\mathcal{D}^{\Sigma}_J(P) \geq \mathcal{D}_J(P \circ \mathcal{K}_\top)$, and $\mathcal{D}^{\max}_J(P) \geq \mathcal{D}_J(P \circ \mathcal{K}_\top)$. If either $\mathcal{D}^{\Sigma}_J(P) \neq \infty$ or $\mathcal{D}^{\max}_J(P) \neq \infty$, then for all $\mathcal{K}_i, \mathcal{K}_j \in P$, we get $J(\mathsf{Weakest}(\mathcal{K}_i, \mathcal{K}_j)) \neq 0$. As J is an information measure, we have that $\mathsf{Weakest}(\mathcal{K}_i, \mathcal{K}_j) \neq [\top]$, for all $\mathcal{K}_i, \mathcal{K}_j \in P$. This implies that $[\top] \notin \mathbb{SI}(\mathcal{K}_i, \mathcal{K}_j)$, for all $\mathcal{K}_i, \mathcal{K}_j \in P$. Thus, there is no inconsistent knowledge base in P, which implies that $\mathcal{K}_i \cup \mathcal{K}_\top$ is consistent, for all $\mathcal{K}_i \in P$. Therefore, $\mathcal{D}_J(\mathcal{K}_i, \mathcal{K}_\top) = \mathcal{D}_J(\mathcal{K}_\top, \mathcal{K}_i) = 0$. Thus,

$$\left(\sum_{\mathcal{K}_i \in P} \mathcal{D}_J(\mathcal{K}_i, \mathcal{K}_\top)\right) + \left(\sum_{\mathcal{K}_i \in P} \mathcal{D}_J(\mathcal{K}_\top, \mathcal{K}_i)\right) = 0,$$

and $max\{\mathcal{D}_J(\mathcal{K}_i, \mathcal{K}_\top) \mid \mathcal{K}_i \in P\} = max\{\mathcal{D}_J(\mathcal{K}_\top, \mathcal{K}_i) \mid \mathcal{K}_i \in P\} = 0$. Therefore, $\mathcal{D}_J^{\Sigma}(P) = \mathcal{D}_J^{\Sigma}(P \circ \mathcal{K}_\top)$ and $\mathcal{D}_J^{\max}(P) = \mathcal{D}_J^{\max}(P \circ \mathcal{K}_\top)$, which means that $\mathcal{D}_J^{\Sigma}(P) \geq \mathcal{D}_J^{\Sigma}(P \circ \mathcal{K}_\top)$ and $\mathcal{D}_J^{\max}(P) \geq \mathcal{D}_J^{\max}(P \circ \mathcal{K}_\top)$.

As for properties CO, MAJ and MAJL observe that for every function $\mathcal{D} : \mathbb{K} \times \mathbb{K} \rightarrow \mathbb{R}^{\infty}_{\geq 0}$, every knowledge base profile P, and knowledge base $\mathcal{K}$ we have that

$$\mathcal{D}^{\Sigma}(P) \leq \mathcal{D}^{\Sigma}(P \circ \mathcal{K})$$
$$\mathcal{D}^{\max}(P) \leq \mathcal{D}^{\max}(P \circ \mathcal{K})$$

as both sum (given non-negative summands) and max are monotonic. Therefore, CO is satisfied, whereas neither MAJ and MAJL is satisfied for any of the considered functions. □