Measuring Inconsistency with Many-Valued Logics

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Abstract

We present a general scheme for inconsistency measurement that generalizes previously proposed approaches based on many-valued logics. We also develop novel instantiations of this schema based on fuzzy logic and investigate their compliance with several rationality postulates for inconsistency measurement, their expressivity, and their computational complexity.

Keywords: inconsistency measure, many-valued logics, fuzzy logic

1. Introduction

A general challenge in knowledge representation and reasoning is the handling of inconsistent information. A quantitative treatment of this challenge is provided by the field of *inconsistency measurement*, see e. g. (Hunter and Konieczny, 2004; Grant and Hunter, 2006) for some early surveys. In this field, the main object of research is the *inconsistency measure*, i. e., a function that assigns a non-negative real value to a knowledge base with the informal meaning that larger values indicate a larger degree of inconsistency. These kind of measures are useful for the tasks of identifying the culprits of inconsistency (Hunter and Konieczny, 2010), as well as manual and automatic debugging of knowledge bases (Grant and Hunter, 2011; Potyka and Thimm, 2014) and inconsistent-tolerant reasoning (Potyka and Thimm, 2015). The traditional setting for inconsistency measurement is that of classical propositional logic and, beginning with Knight's inconsistency measure from (Knight, 2001), a lot of proposals of inconsistency measures have been made for this setting, see e. g. (Grant and Hunter, 2016; McAreavey et al., 2014; Jabbour et al., 2014, 2015; Ammoura et al., 2015; Thimm, 2016c) for some recent works.

As the magnitude and diversity of these different measures show, the problem of conceptualizing a quantitative notion of inconsistency is not a trivial one. A common approach to categorize inconsistency measures is by differentiating whether they operate on the *formula-level* or on the *language-level*. The former category is also called the *syntactic approach* while the latter is called the *semantic approach* (Hunter and Konieczny, 2004). Measures belonging to the syntactic approach usually make use of *minimal inconsistent subsets*, i. e., subsets of the knowledge base that are inconsistent but removing any formula renders them consistent. A simple approach for measuring inconsistency is then to simply take the *number* of minimal inconsistent subsets of a knowledge base as the value of inconsistency, cf. (Hunter and Konieczny, 2008). More recent measures also take the relationships between minimal inconsistent subsets into account (Jabbour and Sais, 2016). Other measures belonging to the syntactic approach may exploit other notions such as maximal consistent subsets (Ammoura et al., 2015) but the commonality of these approaches is that they focus on conflicts between formulas of the knowledge base. On the other hand, measures belonging to the semantic approach focus on conflicts between language components. More precisely, these measures aim at identifying those atoms of the underlying language that are conflicting and they usually employ non-classical and many-valued logics as a tool for that. While the class of syntactic measures, in particular those based on minimal inconsistent subsets, has received a lot of attention in the recent literature (Ammoura et al., 2015; Jabbour et al., 2016; Jabbour and Sais, 2016; Ammoura et al., 2017), the class of semantic measures, in particular those based on many-valued logics, is still in need for a deeper investigation and categorization.

In this paper, we are interested in approaches to inconsistency measurement that—in one form or the other—employ many-valued logics. For example, the contension measure from (Grant and Hunter, 2011) seeks three-valued interpretations that minimize the use of the third non-classical truth value and uses this number as the inconsistency value (note that similar versions of this measure appear in other papers). Another example (which is usually even not regarded a *semantic* measure) is Knight's inconsistency measure (Knight, 2001) which seeks probability functions that maximize the probabilities of all formulas of the knowledge base and, basically, uses this value as the inconsistency value. We generalize these measures by developing an abstract scheme based on evaluation functions of arbitrary interpretations and an optimization problem, and show that previous approaches based on many-valued logics are subsumed by this scheme. Furthermore, we develop novel instantiations of this scheme based on fuzzy logic (Hájek, 1998) that are parametrized by the use of different T-norms and T-conorms, i. e., fuzzy generalizations of classical conjunction and disjunction. More precisely, the contributions of this paper are as follows:

1. We propose a general scheme for inconsistency measurement that subsumes many existing measures based on many-valued logics (Section 3) and analyse its general properties (Section 6).
2. We present a family of inconsistency measures based on fuzzy logic that instantiates this general schema and investigate their properties (Section 4). In particular:
   (a) We evaluate the compliance of these measures wrt. 18 rationality postulates from (Hunter and Konieczny, 2006; Thimm, 2009; Hunter and Konieczny, 2010; Mu et al., 2011a,b; Thimm, 2013; Besnard, 2014) (Section 5.1).
   (b) We evaluate the expressiveness of these measures and show that they are maximally expressive (Section 5.2).
   (c) We determine the computational complexity of common tasks involving inconsistency measures and show that they reside on the first level of the polynomial hierarchy (Section 5.3).

Furthermore, Section 2 recalls necessary preliminaries, Section 7 discusses related works, and Section 8 concludes the paper with a summary. Proofs of technical results can be found in Appendix A.

2. Preliminaries

Let At be some fixed propositional signature, i. e., a (possibly infinite) set of propositions, and let $\mathcal{L}$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*and*), $\vee$ (*or*), and $\neg$ (*negation*).

Definition 1. A knowledge base $\mathcal{K}$ is a finite set of formulas $\mathcal{K} \subseteq \mathcal{L}$. Let $\mathbb{K}(\mathsf{At})$ be the set of all knowledge bases, i. e., the power set of $\mathcal{L}$.

We write $\mathbb{K}$ instead of $\mathbb{K}(\mathsf{At})$ when there is no ambiguity regarding the signature. If X is a formula or a set of formulas we write $\mathsf{At}(X)$ to denote the set of propositions appearing in X. Semantics to a propositional language is given by *interpretations* and an *interpretation* ω on At is a function $\omega : \mathsf{At} \to \{\mathsf{true}, \mathsf{false}\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At. An interpretation ω *satisfies* (or is a *model* of) a proposition $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \mathsf{true}$. The satisfaction relation $\models$ is extended to formulas as usual.

For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$. Define the set of models $\mathsf{Mod}(X) = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models X\}$ for every formula or set of formulas X. A formula or set of formulas X_1 *entails* another formula or set of formulas X_2, denoted by $X_1 \models X_2$, if $\mathsf{Mod}(X_1) \subseteq \mathsf{Mod}(X_2)$. Two formulas or sets of formulas X_1, X_2 are *equivalent*, denoted by $X_1 \equiv X_2$, if $\mathsf{Mod}(X_1) = \mathsf{Mod}(X_2)$. Furthermore, two sets of formulas X_1, X_2 are *semi-extensionally equivalent* if there is a bijection $s : X_1 \rightarrow X_2$ such that for all $\alpha \in X_1$ we have $\alpha \equiv s(\alpha)$ (Thimm, 2013). We denote this by $X_1 \equiv^s X_2$. If $\mathsf{Mod}(X) = \emptyset$ we also write $X \models\perp$ and say that X is *inconsistent*.

Let $\mathbb{R}^{\infty}_{\geq 0}$ be the set of non-negative real values including ∞. Inconsistency measures are functions $\mathcal{I} : \mathbb{K} \rightarrow \mathbb{R}^{\infty}_{\geq 0}$ that aim at assessing the severity of the inconsistency in a knowledge base $\mathcal{K}$. The basic idea is that the larger the inconsistency in $\mathcal{K}$ the larger the value $\mathcal{I}(\mathcal{K})$, where an inconsistency value 0 characterizes a consistent knowledge base, cf. (Hunter and Konieczny, 2006). Inconsistency is a concept that is not easily quantified but there have been several proposals for inconsistency measures so far, in particular for classical propositional logic, see e. g. (Grant and Hunter, 2016; McAreavey et al., 2014; Jabbour et al., 2014, 2015; Ammoura et al., 2015; Thimm, 2016c) for some recent works.

One of the first approaches to measure inconsistency is Knight's measure $\mathcal{I}_\eta$, which is based on probability functions over the underlying propositional language (Knight, 2001). A *probability function* P on $\mathcal{L}$ is a function $P : \Omega(\mathsf{At}) \rightarrow [0, 1]$ with $\sum_{\omega \in \Omega(\mathsf{At})} P(\omega) = 1$. We extend P to assign a probability to any formula $\phi \in \mathcal{L}(\mathsf{At})$ by defining

$$P(\phi) = \sum_{\omega \models \phi} P(\omega)$$

Let $\mathcal{P}(\mathsf{At})$ be the set of all those probability functions. The idea of (Knight, 2001) is to seek a probability function that maximizes the probability of each formula of a knowledge base $\mathcal{K}$. Therefore, the smaller the maximal probability that can be assigned to all formulas the more inconsistent the knowledge base.

Definition 2. The *η-inconsistency measure* $\mathcal{I}_\eta : \mathbb{K} \rightarrow \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_\eta(\mathcal{K}) = 1 - \max\{\xi \mid \exists P \in \mathcal{P}(\mathsf{At}) : \forall \alpha \in \mathcal{K} : P(\alpha) \geq \xi\}$$

for $\mathcal{K} \in \mathbb{K}$.

Note that we modified the definition of $\mathcal{I}_\eta$ slightly compared to the original definition in order to ensure that consistent knowledge bases receive an inconsistency value of zero. The original definition only consists of the term $\max\{\ldots\}$ and thus attained 1 for consistency and 0 for maximal inconsistency.

The contension measure $\mathcal{I}_c$ (Grant and Hunter, 2011) utilizes three-valued interpretations for propositional logic (Priest, 1979).[1] A three-valued interpretation υ on At is a function $\upsilon : \mathsf{At} \to \{T, F, B\}$ where the values T and F correspond to the classical true and false, respectively. The additional truth value B stands for *both* and is meant to represent a conflicting truth value for a proposition. Taking into account the *truth order* $\prec$ defined via $F \prec B \prec T$, an interpretation υ is extended to arbitrary formulas via $\upsilon(\phi_1 \wedge \phi_2) = \min_\prec(\upsilon(\phi_1), \upsilon(\phi_2))$, $\upsilon(\phi_1 \vee \phi_2) = \max_\prec(\upsilon(\phi_1), \upsilon(\phi_2))$, and $\upsilon(\neg T) = F$, $\upsilon(\neg F) = T$, $\upsilon(\neg B) = B$. Then, an interpretation υ satisfies a formula α, denoted by $\upsilon \models^3 \alpha$ if either $\upsilon(\alpha) = T$ or $\upsilon(\alpha) = B$. Then inconsistency can be measured by seeking an interpretation υ that assigns B to a minimal number of propositions.

Definition 3. The *contension inconsistency measure* $\mathcal{I}_c : \mathbb{K} \to \mathbb{R}^\infty_{\geq 0}$ is defined as

$$\mathcal{I}_c(\mathcal{K}) = \min\{|\upsilon^{-1}(B)| \mid \upsilon \models^3 \mathcal{K}\}$$

for $\mathcal{K} \in \mathbb{K}$.

We conclude this section with a small example illustrating the measures introduced above.

Example 1. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be given as

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\} \qquad \mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

Consider the probability function $P_1 \in \mathcal{P}(\{a, b, c, d\})$ as defined in Table 1. Here we have

$$P_1(a) = P_1(\neg a \wedge \neg b) = 0.5, \text{ and}$$
$$P_1(b \vee c) = P_1(d) = 1,$$

and thus $P_1(\phi) \geq 0.5$ for all $\phi \in \mathcal{K}_1$. Furthermore, there can be no other P' that assigns larger probability to all $\phi \in \mathcal{K}_1$. Hence, we have $\mathcal{I}_\eta(\mathcal{K}_1) = 1 - 0.5 = 0.5$. The function P_1 can also be used to determine $\mathcal{I}_\eta(\mathcal{K}_2) = 0.5$.

Furthermore, consider $\upsilon_1 : \{a, b, c, d\} \to \{T, F, B\}$ defined via

$$\upsilon_1(a) = B \qquad \upsilon_1(b) = F \qquad \upsilon_1(c) = \upsilon_1(d) = T$$

[1] Note that similar formalizations of this idea have been given in (Hunter and Konieczny, 2010; Ma et al., 2007, 2011).

$\omega \in \Omega(\{a,b,c,d\})$	$\omega(a)$	$\omega(b)$	$\omega(c)$	$\omega(d)$	$P_1(\omega)$
ω_1	false	false	false	false	0
ω_2	false	false	false	true	0
ω_3	false	false	true	false	0
ω_4	false	false	true	true	0.5
ω_5	false	true	false	false	0
ω_6	false	true	false	true	0
ω_7	false	true	true	false	0
ω_8	false	true	true	true	0
ω_9	true	false	false	false	0
ω_{10}	true	false	false	true	0
ω_{11}	true	false	true	false	0
ω_{12}	true	false	true	true	0
ω_{13}	true	true	false	false	0
ω_{14}	true	true	false	true	0
ω_{15}	true	true	true	false	0
ω_{16}	true	true	true	true	0.5

Table 1: Probability function P_1 from Example 1.

Then $\upsilon_1 \models^3 \phi$ for all $\phi \in \mathcal{K}_1$ and there is no other υ' that assigns B to fewer propositions, yielding $\mathcal{I}_c(\mathcal{K}_1) = 1$. For $\upsilon_2 : \{a, b\} \to \{T, F, B\}$ defined via

$$\upsilon_2(a) = \upsilon_2(b) = B$$

we have $\upsilon_2 \models^3 \phi$ for all $\phi \in \mathcal{K}_2$ and there is no other υ' that assigns B to fewer propositions, yielding $\mathcal{I}_c(\mathcal{K}_2) = 2$.

3. A General Scheme for Inconsistency Measurement

A common feature of the measures $\mathcal{I}_\eta$ and $\mathcal{I}_c$ is that they both employ a many-valued logic for the task of measuring inconsistency. The measure $\mathcal{I}_\eta$ uses probability functions, which essentially map formulas to values in $[0, 1]$, and $\mathcal{I}_c$ uses 3-valued interpretations, which map formulas to values in $\{T, F, B\}$. In the following, we generalized this idea to *arbitrary* many-valued logics. For that, we use the following general definition for interpretations.

Definition 4. Let S be any set (the "truth" values). A function $\omega : \mathcal{L} \to S$ is called an *S-interpretation function*.

This definition is general enough to subsume a wide variety of many-valued interpretations, such as probability functions and three-valued interpretations.

Example 2. Let $S_U = [0, 1]$ and $\omega_P : \mathcal{L} \to S_U$ be any function satisfying the following conditions:

1. $\omega_P(\top) = 1$
2. $\omega_P(\alpha \vee \beta) = \omega_P(\alpha) + \omega_P(\beta)$ if $\alpha \wedge \beta \models \perp$

Then ω_P is equivalent to a probability function, i. e., there is $Q \in \mathcal{P}(\mathsf{At})$ with $\omega_P(\alpha) = Q(\alpha)$ for all $\alpha \in \mathcal{L}$. Let Ω_P be the set of all such functions ω_P.

Example 3. Let $S_3 = \{F, T, B\}$. Let ω_3 be any function satisfying the following conditions (using again the order $F \prec B \prec T$):

1. $\omega_3(\top) = T$
2. $\omega_3(\alpha \wedge \beta) = \min_\prec(\omega_3(\alpha), \omega_3(\beta))$
3. $\omega_3(\neg\alpha) = T$ if $\omega_3(\alpha) = F$, $\omega_3(\neg\alpha) = F$ if $\omega_3(\alpha) = T$, otherwise $\omega_3(\alpha) = B$

Then ω_3 is equivalent to a three-valued interpretation of Priest's logic (Priest, 1979). Let Ω_3 be the set of all such interpretations.

The next commonality of $\mathcal{I}_\eta$ and $\mathcal{I}_c$ is the numerical evaluation of an interpretation wrt. the given knowledge base $\mathcal{K}$. For $\mathcal{I}_\eta$, an interpretation (=probability function) is evaluated by using the minimal probability among all formulas of $\mathcal{K}$ (which is equivalent to the maximal value s. t. all formulas have at least this probability). For $\mathcal{I}_c$, an interpretation is either dismissed (in case at least one formula of $\mathcal{K}$ is assigned the value F), or the number of atoms assigned the value B is used as the evaluation. We generalize this using the following definition.

Definition 5. Let Ω be a set of S-interpretations. An *S-evaluation function E* is a function $E : \mathbb{K} \times \Omega \to \mathbb{R}^{\infty}_{\geq 0}$.

In other words, given a knowledge base $\mathcal{K}$ and an S-interpretation ω, the value $E(\mathcal{K}, \omega)$ represents the evaluation of ω wrt. $\mathcal{K}$.

Finally, the evaluations of all interpretations are compared and the optimal value is chosen.

Definition 6. Let Ω be a set of S-interpretations, E an S-evaluation function, and $\mathcal{K} \in \mathbb{K}$. Define $\mathcal{I}_{\Omega,E} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ via

$$\mathcal{I}_{\Omega,E}(\mathcal{K}) = \min\{E(\mathcal{K}, \omega) \mid \omega \in \Omega\}$$

The measure $\mathcal{I}_{\Omega,E}$ represents a general scheme for instantiating a multitude of different inconsistency measures, based on the selection of Ω and E. Given any set Ω of interpretations, the inconsistency value of a knowledge base $\mathcal{K}$ is determined by the minimal evaluation of an interpretation in Ω wrt. E.

As special cases, this scheme subsumes the measures $\mathcal{I}_\eta$ and $\mathcal{I}_c$. For that, consider first the evaluation function $E_{\min}$ defined for S_U-interpretation functions.

Definition 7. Let Ω_U be a set of S_U-interpretation functions. Define $E_{\min} : \mathbb{K} \times \Omega_U \to \mathbb{R}^{\infty}_{\geq 0}$ via

$$E_{\min}(\mathcal{K}, \omega) = 1 - \min\{\omega(\alpha) \mid \alpha \in \mathcal{K}\}$$

Using this evaluation function, we can show that our general scheme subsumes $\mathcal{I}_\eta$.

Proposition 1. $\mathcal{I}_{\Omega_P,E_{\min}} = \mathcal{I}_\eta$.

Consider now the evaluation function $E_{\#}$ defined for Ω_3.

Definition 8. Define $E_{\#} : \mathbb{K} \times \Omega_3 \to \mathbb{R}^{\infty}_{\geq 0}$ via

$$E_{\#}(\mathcal{K}, \omega) = \begin{cases} |\{a \in \mathsf{At} \mid \omega(a) = B\}| & \text{if } \omega(\alpha) \in \{T, B\} \\ & \text{for all } \alpha \in \mathcal{K} \\ \infty & \text{otherwise} \end{cases}$$

Proposition 2. $\mathcal{I}_{\Omega_3, E_{\#}} = \mathcal{I}_c$.

Propositions 1 and 2 show that $\mathcal{I}_\eta$ and $\mathcal{I}_c$ are special cases of our general scheme $\mathcal{I}_{\Omega,E}$. We will look at some further measures from the literature in Section 7 and continue first with the presentation of a novel measure based on the scheme $\mathcal{I}_{\Omega,E}$.

4. Instantiations based on Fuzzy Logic

The measure $\mathcal{I}_{\Omega,E}$ allows the development of a broad class of novel inconsistency measures by appropriately instantiating Ω and E. The previous section already showed that $\mathcal{I}_\eta$ and $\mathcal{I}_c$ give two *meaningful* instantiations of this scheme, but it is also clear that not every instantiation necessarily leads to an inconsistency measure. For example, consider again the set Ω_3 of three-valued interpretations but with the evaluation function $E'_{\#}$ defined via $E'_{\#}(\mathcal{K}, \omega) = |\{\alpha \in \mathcal{K} \mid \omega(\alpha) = B\}|$, which uses the number of formulas in $\mathcal{K}$ assigned the value B. Then the measure $\mathcal{I}_{\Omega_3, E'_{\#}}$ does not even satisfy the basic property of inconsistency measures that only consistent knowledge bases receive an inconsistency value of zero, as e. g. $\mathcal{I}_{\Omega_3, E'_{\#}}(\{a, \neg a\}) = 0$ (define ω via $\omega(a) = T$, then $\omega(a) = T$ and $\omega(\neg a) = F$).[2] However, in the remainder of this section we develop a more appropriate novel instantiation—more precisely, a novel *family* of instantiations—of the general scheme based on fuzzy logic (Hájek, 1998).

Similarly to probability theory, fuzzy logic can be seen as an extension of classical logic with many truth values, where formulas are assigned truth values in the unit interval $[0, 1]$ instead of the classical values true (= 1) and false (= 0). In difference to probability theory, fuzzy logic does not utilize an *intensional* evaluation of truth values—by using normalized probability functions—but an *extensional* evaluation using operators representing fuzzified extensions of the logical connectors conjunction, disjunction, and negation.[3]

[2]In fact, every knowledge base has an inconsistency value 0 wrt. $\mathcal{I}_{\Omega_3, E'_{\#}}$.

[3]Put simply, an extensional evaluation refers to evaluations of compound formulas using their

For the purpose of this paper we use the following definition of a T-norm, which is the fuzzy version of the logical conjunction.

Definition 9. A function $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *restricted T-norm* if it satisfies for every $x, y, z \in [0, 1]$

1. $t(x, y) = t(y, x)$ (*commutativity*)
2. $t(x, t(y, z)) = t(t(x, y), z)$ (*associativity*)
3. if $x \leq y$ then $t(x, z) \leq t(y, z)$ (*monotony*)
4. $t(x, 1) = x$ (*neutrality*)
5. $t(x, y) = 0$ iff $x = 0$ or $y = 0$

Note that the usual definition of a T-norm only considers properties 1–4 from above, cf. (Hájek, 1998). *Commutativity* and *associativity* represent the basic property that the order of components in a conjunction is irrelevant. *Monotony* states that increasing the truth value of one component cannot decrease the truth value of the whole conjunction. *Neutrality* states that conjunctively combining any truth value with 1 ($=$ $\mathtt{true}$) simplifies to this truth value. We also demand property 5 to be satisfied in order to obtain well-behaved inconsistency measures (see below). In fact, property 5 is the inverse of the property *nil-potency* which is sometimes considered for T-norms and demands $t(x, (t(x, \ldots, t(x, x) \ldots) = 0$ for every $x \in (0, 1)$ and a finite application of t. Observe that any T-norm according to Definition 9 properly extends classical conjunction as for the classical values 0 and 1 we have $t(1, 1) = 1$ and $t(1, 0) = t(0, 1) = t(0, 0) = 0$. For the remainder of this paper we will call *restricted T-norms* simply *T-norms* but be reminded that we always assume property 5 to hold as well.

As a T-norm t is commutative and associative the generalization to arbitrary many arguments is well-defined. Therefore, we abbreviate for $X = \{x_1, \ldots, x_n\}$ with $x_1, \ldots, x_n \in [0, 1]$

$$t(X) = t(x_1, \ldots, x_n) = t(x_1, t(x_2, \ldots t(x_{n-1}, x_n) \ldots))$$

Noteworthy examples of T-norms are the following

$$t_{\text{min}}(x, y) = \min\{x, y\}$$
$$t_{\text{prod}}(x, y) = xy$$

components; as e. g. the probability $P(\alpha \wedge \beta)$ cannot be represented as a combination of the probabilities $P(\alpha)$ and $P(\beta)$ (in general), probability theory cannot be phrased using extensional evaluations, cf. (Hailparin, 1984).

$t_{\min}$ is called the minimum-norm and t_{prod} is the product-norm. Note that both functions comply with the properties 1–5 from Definition 9. Another function that is often used in the context of fuzzy logic is the Łukasiewicz-norm $t_{Ł}$ defined via $t_{Ł}(x, y) = \max\{x + y - 1, 0\}$. Note that $t_{Ł}$ violates property 5 as e. g. $t_{Ł}(0.5, 0.5) = 0$ and is therefore not considered in the remainder of this paper.

The counterpart to T-norms are *T-conorms* (sometimes also called S-norms) which generalize classical disjunction and are defined as follows.

Definition 10. A function $s : [0, 1] \times [0, 1] \to [0, 1]$ is called a *restricted T-conorm* if it satisfies for every $x, y, z \in [0, 1]$

1. $s(x, y) = s(y, x)$ (*commutativity*)
2. $s(x, s(y, z)) = s(s(x, y), z)$ (*associativity*)
3. if $x \leq y$ then $s(x, z) \leq s(y, z)$ (*monotony*)
4. $s(x, 0) = x$ (*neutrality*)
5. $s(x, y) = 1$ iff $x = 1$ or $y = 1$

Properties 1–5 are analogous to properties 1–5 of Definition 9 (note however the different neutral element in item 4) and property 5 is again non-standard and added to obtain well-behaved inconsistency measures (see below). Observe that any T-conorm according to Definition 10 properly extends classical disjunction as for the classical values 0 and 1 we have $s(1, 1) = s(1, 0) = s(0, 1) = 1$ and $s(0, 0) = 0$. Again, for the remainder of this paper we will call *restricted T-conorms* simply *T-conorms* but be reminded that we always assume property 5 to hold as well.

Noteworthy examples of T-conorms are the following

$$s_{\max}(x, y) = \max\{x, y\}$$
$$s_{\text{psum}}(x, y) = x + y - xy$$

$s_{\max}$ is called the maximum-conorm and s_{psum} is called the probabilistic sum. Note that both functions comply with the properties 1–5 from Definition 10.

For negation we only consider the classical idempotent negation n defined via $n(x) = 1 - x$.

T-norms and T-conorms are related in a similar way as classical conjunction and disjunction are related through the De Morgan rules. More precisely, a T-conorm s is the *dual* of a T-norm t if $s(x, y) = n(t(n(x), n(y)))$. Observe that $s_{\max}$ is the dual of $t_{\min}$ and that s_{psum} is the dual of t_{prod}.

Definition 11. Let t be any T-norm and s any T-conorm. A *fuzzy interpretation* $\omega_{t,s}$ for t and s is an S_U-interpretation function $\omega_{t,s} : \mathcal{L} \to [0,1]$ which satisfies

1. $\omega_{t,s}(\neg\alpha) = n(\omega_{t,s}(\alpha))$
2. $\omega_{t,s}(\alpha \wedge \beta) = t(\omega_{t,s}(\alpha), \omega_{t,s}(\beta))$
3. $\omega_{t,s}(\alpha \vee \beta) = s(\omega_{t,s}(\alpha), \omega_{t,s}(\beta))$

For a T-norm t and a T-conorm s let $\Omega_{t,s}$ denote the set of fuzzy interpretations for t and s. If s is the dual of t we simply write Ω_t.

We now follow a similar approach as for the η-inconsistency measure to define an inconsistency measure based on fuzzy evaluation functions.

Definition 12. Let t be any T-norm and s any T-conorm. Define the function $E_t^{\text{fuz}} : \mathbb{K} \times \Omega_{t,s} \to \mathbb{R}^{\infty}_{\geq 0}$ via

$$E_t^{\text{fuz}}(\mathcal{K}, \omega) = n(t(\{\omega(\alpha) \mid \alpha \in \mathcal{K}\}))$$

Then the function $\mathcal{I}_{t,s}^{\text{fuz}} = \mathcal{I}_{\Omega_{t,s}, E_t^{\text{fuz}}}$ is called *fuzzy inconsistency measure* based on t and s. We abbreviate $\mathcal{I}_{t,s}^{\text{fuz}}$ by $\mathcal{I}_t^{\text{fuz}}$ if s is the dual of t.

The basic idea of the definition above is as follows. Given some T-norm and T-conorm, all fuzzy interpretations for the given propositional language are considered. Each fuzzy interpretation assigns a value in $[0,1]$ to each formula of the given knowledge base and we use the chosen T-norm to combine these values into a single value, which can be interpreted as the fuzzy truth value of the whole knowledge base (wrt. the interpretation). For normalization purposes we invert this value (using the fuzzy negation) so that larger values mean less truth. Finally, we consider this evaluation for all fuzzy interpretations and select the minimum as the final inconsistency value.

Example 4. We continue Example 1 and consider $\mathcal{K}_1$ and $\mathcal{K}_2$ given as

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\} \qquad \mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

Consider the T-norm t_{prod} and the fuzzy interpretation $\omega \in \Omega_{t_{\text{prod}}}$ defined via (note that we use the dual s_{psum} of t_{prod} as T-conorm)

$$\omega(a) = 0.3 \qquad \omega(b) = 0.7 \qquad \omega(c) = 0.5 \qquad \omega(d) = 1$$

and $\omega(\alpha)$ for arbitrary formulas $\alpha \in \mathcal{L}$ is defined as prescribed in Definition 11. For the formulas in $\mathcal{K}_1$ we obtain

$$\begin{aligned}
\omega(a) &= 0.3 \\
\omega(b \vee c) &= s_{\text{psum}}(\omega(b), \omega(c)) = s_{\text{psum}}(0.7, 0.5) \\
&= 0.7 + 0.5 - 0.7 \cdot 0.5 = 0.85 \\
\omega(\neg a \wedge \neg b) &= t_{\text{prod}}(n(\omega(a)), n(\omega(b))) \\
&= (1 - 0.3)(1 - 0.7) = 0.21 \\
\omega(d) &= 1
\end{aligned}$$

and therefore

$$\begin{aligned}
E^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}_1, \omega) &= n(t_{\text{prod}}(\{0.3, 0.85, 0.21, 1\})) \\
&= 1 - (0.3 \cdot 0.85 \cdot 0.21 \cdot 1) = 0.94645
\end{aligned}$$

However, note that $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}_1) = 0.75$ by considering $\omega' \in \Omega_{t_{\text{prod}}}$ with $\omega'(a) = 0.5$, $\omega'(c) = \omega'(d) = 1$ and $\omega'(b) = 0$. Furthermore, we obtain $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}_2) = 0.9375$.

Definition 12 gives rise to a variety of inconsistency measures, depending on the chosen T-norm and T-conorm. Observe that using $t_{\min}$ is not so interesting after all:

Proposition 3.

$$\mathcal{I}^{\mathit{fuz}}_{t_{\min}}(\mathcal{K}) = \begin{cases} 1/2 & \textit{if } \mathcal{K} \models \perp \\ 0 & \textit{otherwise} \end{cases}$$

So $\mathcal{I}^{\text{fuz}}_{t_{\min}}$ is equivalent to a SAT test and (up to normalization) to the drastic inconsistency measure $\mathcal{I}_d$ (Hunter and Konieczny, 2006) which uses 1 as the value for inconsistent knowledge bases. Note that Proposition 3 is not true for arbitrary T-norms and T-conorms as shown in Example 4.

Our general scheme for inconsistency measures from the previous section allows the definition of a multitude of variants of $\mathcal{I}^{\text{fuz}}_{t,s}$. As an example, we also consider one other instantiation as follows.

Definition 13. Let t be any T-norm and s any T-conorm. Define the function $E^{\text{fuz}}_{\Sigma} : \mathbb{K} \times \Omega_{t,s} \to \mathbb{R}^{\infty}_{\geq 0}$ via

$$E^{\text{fuz}}_{\Sigma}(\mathcal{K}, \omega) = \sum_{\alpha \in \mathcal{K}} n(\omega(\alpha))$$

Then the function $\mathcal{I}^{\text{fuz},\Sigma}_{t,s} = \mathcal{I}_{\Omega_{t,s},E^{\text{fuz}}_{\Sigma}}$ is called Σ-*fuzzy inconsistency measure* based on t and s. We abbreviate $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$ by $\mathcal{I}^{\text{fuz},\Sigma}_{t}$ if s is the dual of t.

In difference to $\mathcal{I}^{\text{fuz}}_{t,s}$ the measure $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$ aggregates the (negated) fuzzy values of the formulas by summation rather then by using fuzzy negation and the T-norm.

Example 5. We continue Example 4 and consider again $\mathcal{K}_1$ and $\mathcal{K}_2$ given as

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\} \qquad \mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

For ω as defined in Example 4 via

$$\omega(a) = 0.3 \qquad \omega(b) = 0.7 \qquad \omega(c) = 0.5 \qquad \omega(d) = 1$$

and again using t_{prod} and s_{psum} we obtain

$$\begin{aligned} E^{\text{fuz}}_{\Sigma}(\mathcal{K}_1, \omega) &= n(\omega(a)) + n(\omega(b \vee c)) + n(\omega(\neg a \wedge \neg b)) + n(\omega(d)) \\ &= (1 - 0.3) + (1 - 0.85) + (1 - 0.21) + (1 - 1) \\ &= 1.64 \end{aligned}$$

However, note that $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}_1) = 1$ by considering $\omega' \in \Omega_{t_{\text{prod}}}$ with $\omega'(a) = 0.5$, $\omega'(c) = \omega'(d) = 1$ and $\omega'(b) = 0$. Furthermore, we obtain $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}_2) = 2$.

5. Analysis

In this section, we analyze the measures $\mathcal{I}^{\text{fuz}}_{t,s}$ and $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$ by investigating their compliance with rationality postulates (Section 5.1), their expressivity (Section 5.2), and their computational complexity (Section 5.3).

5.1. Rationality Postulates

In the literature, inconsistency measures are usually analytically evaluated using rationality postulates. Starting with (Hunter and Konieczny, 2006) a series of other papers have proposed rationality postulates and argued about their appropriateness, cf. (Besnard, 2014) for a recent discussion.

In order to state the rationality postulates we need some notation. A set $M \subseteq \mathcal{K}$ is called *minimal inconsistent subset* (MI) of $\mathcal{K}$ if $M \models \perp$ and there is no $M' \subset M$ with $M' \models \perp$. Let $\mathsf{MI}(\mathcal{K})$ be the set of all MIs of $\mathcal{K}$.

Definition 14. A formula $\alpha \in \mathcal{K}$ is called *free formula* if $\alpha \notin \bigcup \mathsf{MI}(\mathcal{K})$. Let $\mathsf{Free}(\mathcal{K})$ be the set of all free formulas of $\mathcal{K}$.

In other words, a free formula is basically a formula that is not directly participating in any derivation of a contradiction. Using this definition and the concepts already introduced before, the first five rationality postulates of (Hunter and Konieczny, 2006) can be stated as follows. Let $\mathcal{I}$ be any inconsistency measure, $\mathcal{K}, \mathcal{K}' \in \mathbb{K}$, and $\alpha, \beta \in \mathcal{L}(\mathsf{At})$.

***Consistency* (CO)** $\mathcal{I}(\mathcal{K}) = 0$ if and only if $\mathcal{K}$ is consistent

***Normalization* (NO)** $0 \leq \mathcal{I}(\mathcal{K}) \leq 1$

***Monotony* (MO)** If $\mathcal{K} \subseteq \mathcal{K}'$ then $\mathcal{I}(\mathcal{K}) \leq \mathcal{I}(\mathcal{K}')$

***Free-formula independence* (IN)** If $\alpha \in \mathsf{Free}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$

***Dominance* (DO)** If $\alpha \not\models \perp$ and $\alpha \models \beta$ then $\mathcal{I}(\mathcal{K} \cup \{\alpha\}) \geq \mathcal{I}(\mathcal{K} \cup \{\beta\})$

The first postulate, CO, requires that consistent knowledge bases receive the minimal inconsistency value zero and every inconsistent knowledge base has a strictly positive inconsistency value. This postulate is actually the only generally accepted postulate and describes the minimal requirement for an inconsistency measure. An inconsistency measure $\mathcal{I}$ that satisfies CO does not distinguish between consistent knowledge bases and can, at least, distinguish inconsistent knowledge bases from consistent ones.

The postulate NO states that the inconsistency value is always in the unit interval, thus allowing inconsistency values to be comparable even if knowledge bases are of different sizes. In later works, this postulate is usually regarded as an optional feature.

MO requires that adding formulas to the knowledge base cannot decrease the inconsistency value. Besides CO this is the least disputed postulate and most inconsistency measures do satisfy it.

IN states that removing free formulas from the knowledge base should not change the inconsistency value. The motivation here is that free formulas do not participate in inconsistencies and should not contribute to having a certain inconsistency value.

DO says that substituting a consistent formula α by a weaker formula β should not increase the inconsistency value. Here, as β carries less information than α there should be less opportunities for inconsistencies to occur.

The set of postulates was extended in (Thimm, 2009) for the case of inconsistency measurement in probabilistic logics. However, we can state these postulates also for propositional logic.

Definition 15. A formula $\alpha \in \mathcal{K}$ is called *safe formula* if it is consistent and $\mathsf{At}(\alpha) \cap \mathsf{At}(\mathcal{K} \setminus \{\alpha\}) = \emptyset$. Let $\mathsf{Safe}(\mathcal{K})$ be the set of all safe formulas of $\mathcal{K}$.

A formula is safe, if its signature is disjoint from the signature of the rest of the knowledge base, cf. the concept of language splitting in belief revision (Parikh, 1999; Kourousias and Makinson, 2007). Every safe formula is also a free formula (Thimm, 2009).

***Safe-formula independence* (SI)** If $\alpha \in \mathsf{Safe}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$

***Super-Additivity* (SA)** If $\mathcal{K} \cap \mathcal{K}' = \emptyset$ then $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') \geq \mathcal{I}(\mathcal{K}) + \mathcal{I}(\mathcal{K}')$

***Penalty* (PY)** If $\alpha \notin \mathsf{Free}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) > \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$

The postulate SI requires that removing isolated formulas from a knowledge base cannot change the inconsistency value. This postulate is a weakening of IN, i. e., if a measure $\mathcal{I}$ satisfies IN it also satisfies SI, cf. (Thimm, 2009) and Theorem 1 below.

SA is a strengthening of MO (Thimm, 2009) and requires that the sum of the inconsistency values of two disjoint knowledge bases is not larger than the inconsistency value of the joint knowledge base.

PY is the complementary postulate to IN and states that adding a formula participating in inconsistency must have a positive impact on the inconsistency value.

The following two postulates have been first used in (Hunter and Konieczny, 2010):

***MI-separability* (MI)** If $\mathsf{MI}(\mathcal{K}_1 \cup \mathcal{K}_2) = \mathsf{MI}(\mathcal{K}_1) \cup \mathsf{MI}(\mathcal{K}_2)$ and $\mathsf{MI}(\mathcal{K}_1) \cap \mathsf{MI}(\mathcal{K}_2) = \emptyset$ then $\mathcal{I}(\mathcal{K}_1 \cup \mathcal{K}_2) = \mathcal{I}(\mathcal{K}_1) + \mathcal{I}(\mathcal{K}_2)$

***MI-normalization* (MN)** If $M \in \mathsf{MI}(\mathcal{K})$ then $\mathcal{I}(M) = 1$

MI focuses particularly on the role of minimal inconsistent subsets in the determination of the inconsistency value. It states that the sum of the inconsistency values of two knowledge bases that have “non-interfering” sets of minimal inconsistent subsets should be the same as the inconsistency value of their union.

MN demands that a minimal inconsistent subset is the atomic unit for measuring inconsistency by requiring that the inconsistency value of any minimal inconsistent subset is one.

The following postulates have been proposed in (Mu et al., 2011a) to further define the role of minimal inconsistent subsets in measuring inconsistency:

Attenuation **(AT)** $M, M' \in \mathsf{MI}(\mathcal{K})$ and $\mathcal{I}(M) < \mathcal{I}(M')$ implies $|M| > |M'|$

Equal Conflict **(EC)** $M, M' \in \mathsf{MI}(\mathcal{K})$ and $\mathcal{I}(M) = \mathcal{I}(M')$ implies $|M| = |M'|$

Almost Consistency **(AC)** Let $M_1, M_2, \ldots$ be a sequence of minimal inconsistent sets M_i with $\lim_{i\to\infty} |M_i| = \infty$, then $\lim_{i\to\infty} \mathcal{I}(M_i) = 0$

The postulate AT states that minimal inconsistent sets of smaller size should have a larger inconsistency value. The motivation of this postulate stems from the *lottery paradox* (Kyburg, 1961): Consider a lottery of n tickets and let a_i be the proposition that ticket i, $i = 1, \ldots, n$ will win. It is known that exactly one ticket will win ($a_1 \vee \ldots \vee a_n$) but each ticket owner assumes that his ticket will not win ($\neg a_i$, $i = 1, \ldots, n$). For $n = 1000$ it is reasonable for each ticket owner to believe that he will not win but for e. g., $n = 2$ it is not. Therefore larger minimal inconsistent subsets can be regarded less inconsistent than smaller ones.

The postulate EC is the counterpart of AT and requires minimal inconsistent subsets having the same inconsistency value also to have the same size.

AC considers the inconsistency values on arbitrarily large minimal inconsistent subsets in the limit and requires this to be zero.

The following postulates are from (Mu et al., 2011b).

Contradiction **(CD)** $\mathcal{I}(\mathcal{K}) = 1$ if and only if for all $\emptyset \neq \mathcal{K}' \subseteq \mathcal{K}$, $\mathcal{K}' \models\perp$

Free Formula Dilution **(FD)** If $\alpha \in \mathsf{Free}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) \geq \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$

CD is meant as an extension of NO and states that a knowledge base is maximally inconsistent if all non-empty subsets are inconsistent. Note that CD only makes sense if NO is satisfied as well. FD has been introduced in (Mu et al., 2011b) to serve as a weaker version of IN for normalised measures, i. e., measures satisfying NO. For those, it may be the case that adding free formulas decreases the inconsistency value as they measure a "relative" amount of inconsistency.

The following property is from (Thimm, 2013):

Irrelevance of Syntax **(SY)** If $\mathcal{K}_1 \equiv^s \mathcal{K}_2$ then $\mathcal{I}(\mathcal{K}_1) = \mathcal{I}(\mathcal{K}_2)$

SY states that knowledge bases with pairwise equivalent formulas should receive the same inconsistency value.

In (Besnard, 2014) a series of further postulates have been discussed. For our current study, we only consider the following two:

Exchange **(EX)** If $\mathcal{K}' \not\models\perp$ and $\mathcal{K}' \equiv \mathcal{K}''$ then $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') = \mathcal{I}(\mathcal{K} \cup \mathcal{K}'')$

***Adjunction Invariance* (AI)** $\mathcal{I}(\mathcal{K} \cup \{\alpha, \beta\}) = \mathcal{I}(\mathcal{K} \cup \{\alpha \wedge \beta\})$

EX is similar in spirit to SY and demands that exchanging consistent parts of the knowledge base with equivalent ones should not change the inconsistency value.

AI demands that the set notation of knowledge bases should be equivalent to the conjunction of its formulas in terms of inconsistency values. In difference to EX note that AI has no precondition on the consistency of the considered formulas.

Note that not all postulates are independent and that some are incompatible. Some relationships are summarised in the following theorem, see (Thimm, 2016a) for the proof. In the theorem, a statement "A implies B" is meant to be read as "if a measure satisfies A then it satisfies B"; a statement "$\mathrm{A}_1, \ldots, \mathrm{A}_n$ are incompatible" means "there is no measure satisfying $\mathrm{A}_1, \ldots, \mathrm{A}_n$ at the same time".

Theorem 1.

1. *IN implies SI*
2. *IN implies FD*
3. *SA implies MO*
4. *MN and AC are incompatible*
5. *MN and CD are incompatible*
6. *MO implies FD*
7. *MN, MI, and NO are incompatible*
8. *MN, SA, and NO are incompatible*

We obtain the following results regarding compliance of our measures to the introduced rationality postulates.

Theorem 2. *Let t be any T-norm and s any T-conorm. $\mathcal{I}^{fuz}_{t,s}$ satisfies CO, NO, MO, SI, FD, and AI. In general, $\mathcal{I}^{fuz}_{t,s}$ does not satisfy IN, DO, SA, PY, MI, MN, AT, EC, AC, CD, SY, and EX.*

Theorem 3. *Let t be any T-norm and s any T-conorm. $\mathcal{I}^{fuz,\Sigma}_{t,s}$ satisfies CO, MO, SI, SA, and FD. $\mathcal{I}^{fuz,\Sigma}_{t,s}$ does not satisfy NO, IN, DO, PY, MI, MN, AT, EC, AC, CD, SY, EX, and AI.*

The above theorems apply to *every* pair of T-norm t and T-conorm s. So for specific norms, more postulates may be satisfied. In particular, as $\mathcal{I}^{\mathrm{fuz}}_{t_{\min}}$ is equivalent to the drastic inconsistency measure (see Proposition 3) we directly obtain the following result (for proofs see e. g. (Thimm, 2016a)).

	$\mathcal{I}^{\text{fuz}}_{t,s}$	$\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$	$\mathcal{I}^{\text{fuz}}_{t_{\min}}$	$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$	$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$	$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$	$\mathcal{I}_\eta$	$\mathcal{I}_c$
CO	✓	✓	✓	✓	✓	✓	✓	✓
NO	✓		✓	✓			✓	
MO	✓	✓	✓	✓	✓	✓	✓	✓
IN			✓				✓	✓
DO			✓		✓		✓	✓
SI	✓	✓	✓	✓	✓	✓	✓	✓
SA		✓			✓	✓		
PY								
MI								
MN			✓					
AT			✓				✓	
EC							✓	
AC							✓	
CD								
FD	✓	✓	✓	✓	✓	✓	✓	✓
SY			✓		✓		✓	
EX			✓					✓
AI	✓		✓	✓				✓

Table 2: Compliance of the discussed inconsistency measures wrt. rationality postulates

Theorem 4. $\mathcal{I}^{\textit{fuz}}_{t_{\min}}$ *satisfies* CO, NO, MO, IN, DO, SI, MN, AT, FD, SY, EX, *and* AI. $\mathcal{I}^{\textit{fuz}}_{t_{\min}}$ *does not satisfy* SA, PY, MI, EC, AC, *and* CD.

For $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$ we have the same result as for the general measure $\mathcal{I}^{\text{fuz}}_{t,s}$.

Theorem 5. $\mathcal{I}^{\textit{fuz}}_{t_{\textit{prod}}}$ *satisfies* CO, NO, MO, SI, FD, *and* AI. $\mathcal{I}^{\textit{fuz}}_{t_{\textit{prod}}}$ *does not satisfy* IN, DO, SA, PY, MI, MN, AT, EC, AC, CD, SY, *and* EX.

For the two instantiations $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$ and $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$ we obtain the following.

Theorem 6. $\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\min}}$ *satisfies* CO, MO, DO, SI, SA, FD *and* SY. $\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\min}}$ *does not satisfy* NO, IN, PY, MI, MN, AT, EC, AC, CD, EX, *and* AI.

Theorem 7. $\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\textit{prod}}}$ *satisfies* CO, MO, SI, SA, *and* FD. $\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\textit{prod}}}$ *does not satisfy* NO, IN, DO, PY, MI, MN, AT, EC, AC, CD, SY, EX, *and* AI.

Table 2 gives an overview on the compliance of rationality postulates where ✓ indicates satisfaction of a postulate. The table also shows the compliance of the measures $\mathcal{I}_\eta$ and $\mathcal{I}_c$ for comparison, see (Thimm, 2016a) for the corresponding proofs.

Although it seems that our novel measures satisfy a rather little number of postulates, it should be noted that is a usual behavior as there is still no general agreement on what postulates should be satisfied at all (Thimm, 2016a; Besnard, 2014). In fact, comparing our results with the recent survey from (Thimm, 2016a), there is no other measure $\mathcal{I}$ (among those investigated in (Thimm, 2016a)) that satisfies exactly the same or a superset of postulates as the measures from the family $\mathcal{I}^{\text{fuz}}_{t,s}$—with the exception of $\mathcal{I}^{\text{fuz}}_{t_{\min}}$ of course which is equivalent to the drastic measure $\mathcal{I}_d$ also investigated in (Thimm, 2016a)—. While for the family $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$ there is also no measure in (Thimm, 2016a) that satisfies exactly the same set of postulates, there are measures that satisfy a strict superset of postulates. More precisely, for the general measure $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$ and the instantiation $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$, the measures $\mathcal{I}_{nc}$, $\mathcal{I}^{\text{hit}}_d$, $\mathcal{I}^{\Sigma}_d$, $\mathcal{I}_p$, $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$, and $\mathcal{I}_{\mathsf{MI}}$ satisfy a strict superset of postulates.[4] The same is true for our measure $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$ and the measures $\mathcal{I}_{nc}$, $\mathcal{I}^{\text{hit}}_d$, and $\mathcal{I}^{\Sigma}_d$.[5] However, it should be noted that all our measures—with the exception of $\mathcal{I}^{\text{fuz}}_{t_{\min}}$ again—are indeed novel and different in behavior to existing measures.

5.2. Expressivity

Besides rationality postulates, another (complementary) dimension of evaluating an inconsistency measure is its *expressivity* (Thimm, 2016b), that is, the number of different inconsistency values a measure can attain on some certain sets of knowledge bases. This evaluation measure has been proposed in (Thimm, 2016b) in order to be able to distinguish trivial measures such as $\mathcal{I}_d$—which still satisfies a reasonable number of rationality postulates—from more "fine-grained" assessments of inconsistency. In (Thimm, 2016b), four different *expressivity characteristics* are proposed to evaluate the expressivity of inconsistency measures. We will now recall these characteristics and evaluate our measures wrt. those afterwards.

Before defining expressivity characteristics we need some further notation.

[4] Note that (Thimm, 2016a) also reported $\mathcal{I}_{P_m}$ to satisfy a strict superset of these postulates; however, the original publication (Jabbour and Raddaoui, 2013)—which was cited in (Thimm, 2016a) for this result—falsely claimed that $\mathcal{I}_{P_m}$ satisfies CO. However, $\mathcal{I}_{P_m}$ does not comply with this basic property as e. g. for inconsistent $\mathcal{K}_{Pm} = \{a, \neg(a \wedge a)\}$, $\mathcal{I}_{P_m}(\mathcal{K}_{Pm}) = 0$, cf. Definition 2, Proposition 2, and Section 3 in (Jabbour and Raddaoui, 2013).

[5] For $\mathcal{I}^{\text{hit}}_d$ and $\mathcal{I}^{\Sigma}_d$ please refer to Section 7 for the definitions of those measures and links to the original publications; for the other measures please see (Thimm, 2016a).

Definition 16. Let ϕ be a formula. The *length* $\mathsf{l}(\phi)$ of ϕ is recursively defined as

$$\mathsf{l}(\phi) = \begin{cases} 1 & \text{if } \phi \in \mathsf{At} \\ 1 + \mathsf{l}(\phi') & \text{if } \phi = \neg\phi' \\ 1 + \mathsf{l}(\phi_1) + \mathsf{l}(\phi_2) & \text{if } \phi = \phi_1 \wedge \phi_2 \\ 1 + \mathsf{l}(\phi_1) + \mathsf{l}(\phi_2) & \text{if } \phi = \phi_1 \vee \phi_2 \end{cases}$$

Definition 17. Define the following subclasses of the set of all knowledge bases $\mathbb{K}$:

$$\begin{aligned} \mathbb{K}^v(n) &= \{\mathcal{K} \in \mathbb{K} \mid |\mathsf{At}(\mathcal{K})| \leq n\} \\ \mathbb{K}^f(n) &= \{\mathcal{K} \in \mathbb{K} \mid |\mathcal{K}| \leq n\} \\ \mathbb{K}^l(n) &= \{\mathcal{K} \in \mathbb{K} \mid \forall \phi \in \mathcal{K} : \mathsf{l}(\phi) \leq n\} \\ \mathbb{K}^p(n) &= \{\mathcal{K} \in \mathbb{K} \mid \forall \phi \in \mathcal{K} : |\mathsf{At}(\phi)| \leq n\} \end{aligned}$$

Informally speaking, $\mathbb{K}^v(n)$ is the set of all knowledge bases that mention at most n different propositions, $\mathbb{K}^f(n)$ is the set of all knowledge bases that contain at most n formulas, $\mathbb{K}^l(n)$ is the set of all knowledge bases that contain only formulas with maximal length n, and $\mathbb{K}^p(n)$ is the set of all knowledge bases that contain only formulas that mention at most n different propositions each.

Definition 18. Let $\mathcal{I}$ be an inconsistency measure and $n > 0$. Let $\alpha \in \{v, f, l, p\}$. The *$\alpha$-characteristic* $\mathcal{C}^\alpha(\mathcal{I}, n)$ of $\mathcal{I}$ wrt. n is defined as

$$\mathcal{C}^\alpha(\mathcal{I}, n) = |\{\mathcal{I}(\mathcal{K}) \mid \mathcal{K} \in \mathbb{K}^\alpha(n)\}|$$

In other words, $\mathcal{C}^\alpha(\mathcal{I}, n)$ is the number of different inconsistency values $\mathcal{I}$ assigns to knowledge bases from $\mathbb{K}^\alpha(n)$.

Example 6. We recall a result (Thimm, 2016b) regarding the measure $\mathcal{I}_c$ (let $n > 0$):

$$\begin{aligned} \mathcal{C}^v(\mathcal{I}_c, n) &= n + 1 \\ \mathcal{C}^f(\mathcal{I}_c, n) &= \infty \\ \mathcal{C}^l(\mathcal{I}_c, n) &= \infty \\ \mathcal{C}^p(\mathcal{I}_c, n) &= \infty \end{aligned}$$

In particular, $\mathcal{C}^v(\mathcal{I}_c, n) = n + 1$ means that there are only $n + 1$ different inconsistency values of $\mathcal{I}_c$ on knowledge bases which mention at most n propositions (which follows from the fact that $\mathcal{I}_c$ is defined as the number of propositions that are assigned the truth value B by some three-valued interpretation; as there are n propositions we have $0, \ldots, n$ as possible inconsistency values). Furthermore, $\mathcal{C}^f(\mathcal{I}_c, n) = \infty$ basically says that even knowledge bases with only one formula can have an arbitrary inconsistency value wrt. $\mathcal{I}_c$. So, $\mathcal{I}_c$ is *maximally* expressive wrt. the characteristic $\mathcal{C}^f$ (note that $\mathcal{I}_c$ is not maximally expressive wrt. the characteristic $\mathcal{C}^v$). Note also that $\mathcal{C}^l(\mathcal{I}_c, n) = \infty$ is only true for $n > 1$ as every $\mathcal{K} \in \mathbb{K}^l(1)$ is trivially consistent (all formulas have length 1, so there is no negation) and $\mathcal{C}^l(\mathcal{I}_c, 1)$ trivializes to $\mathcal{C}^l(\mathcal{I}_c, 1) = 1$ (this is true for every measure satisfying the postulate CO).

The next two results show that both our measures $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$ and $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$ are maximally expressive wrt. all four expressivity characteristics.

Theorem 8. *For all* $n > 0$, $\mathcal{C}^v(\mathcal{I}^{\textit{fuz}}_{t_{\textit{prod}}}, n) = \mathcal{C}^f(\mathcal{I}^{\textit{fuz}}_{t_{\textit{prod}}}, n) = \mathcal{C}^p(\mathcal{I}^{\textit{fuz}}_{t_{\textit{prod}}}, n) = \infty$. *For all* $n > 1$, $\mathcal{C}^l(\mathcal{I}^{\textit{fuz}}_{t_{\textit{prod}}}, n) = \infty$.

Theorem 9. *For all* $n > 0$, $\mathcal{C}^v(\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\textit{prod}}}, n) = \mathcal{C}^f(\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\textit{prod}}}, n) = \mathcal{C}^p(\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\textit{prod}}}, n) = \infty$. *For all* $n > 1$, $\mathcal{C}^l(\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\textit{prod}}}, n) = \infty$.

As $\mathcal{I}^{\text{fuz}}_{t_{\min}}$ is equivalent to the drastic inconsistency measure, its expressivity is 2 for all characteristics, cf. (Thimm, 2016b). For $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$ we observe maximal expressivity wrt. all characteristics except $\mathcal{C}^f$.

Theorem 10. *For all* $n > 0$, $\mathcal{C}^v(\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\min}}, n) = \infty$, $\mathcal{C}^f(\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\min}}, n) = n + 1$, *and* $\mathcal{C}^p(\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\min}}, n) = \infty$. *For all* $n > 1$, $\mathcal{C}^l(\mathcal{I}^{\textit{fuz},\Sigma}_{t_{\min}}, n) = \infty$.

The above results, in particular Theorems 8 and 9, show that our measures behave very well wrt. expressivity. It should be noted that only one measure investigated in (Thimm, 2016b) is also maximally expressive wrt. all four expressivity characteristics, namely $\mathcal{I}^{\Sigma}_{d}$, cf. the discussion in Section 5.1.[6] Table 3 summarises the results of this section.

5.3. Computational Complexity

Before discussing the computational complexity of our approaches, we briefly recall some background on complexity theory needed for our results, cf. (Papadimitriou, 1994). P is the class of decision problems decidable by a deterministic

[6]Note that also the measure $\mathcal{I}_{P_m}$ has maximal expressivity but is omitted above as it does not comply with the basic property CO, cf. Footnote 4.

	$\mathcal{C}^v(\mathcal{I},n)$	$\mathcal{C}^f(\mathcal{I},n)$	$\mathcal{C}^l(\mathcal{I},n)$	$\mathcal{C}^p(\mathcal{I},n)$
$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$	∞	∞	∞^*	∞
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$	∞	∞	∞^*	∞
$\mathcal{I}^{\text{fuz}}_{t_{\text{min}}}$	2	2	2^*	2
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{min}}}$	∞	$n+1$	∞^*	∞

Table 3: Characteristics of inconsistency measures ($n \geq 1$); *only for $n > 1$

Turing machine in time polynomial wrt. the length of the instance. NP is the class of decision problems decidable by a non-deterministic Turing machine in polynomial time, i. e., those decision problems where a *proof* of a positive instance can be verified in polynomial time. The class coNP is the set of problems D where the complement $\overline{D}$ is in NP (the complement $\overline{D}$ of a problem D is the same as D with positive and negative instances reversed). For two decision problems D, D', the *conjunction* $D \wedge D'$ is the decision problem with positive instances being pairs (x, x') where x is a positive instance of D and x' is a positive instance of D'. Then $\mathsf{D}^{\mathsf{P}}_1$ is the set of all conjunctions $D \wedge D'$ with $D \in \mathsf{NP}$ and $D' \in \mathsf{coNP}$. Finally, a problem D is *complete* for a complexity class $\mathcal{C}$ if every problem $D' \in \mathcal{C}$ can be reduced to D with only polynomial overhead.

Let $\mathcal{I}$ be any inconsistency measure. We consider the following three decision problems for our study of the computational complexity of our measures, taken from (Thimm and Wallner, 2016):

EXACT$_\mathcal{I}$	**Input**:	$\mathcal{K} \in \mathbb{K}$, $x \in \mathbb{R}^{\infty}_{\geq 0}$
	Output:	TRUE iff $\mathcal{I}(\mathcal{K}) = x$
UPPER$_\mathcal{I}$	**Input**:	$\mathcal{K} \in \mathbb{K}$, $x \in \mathbb{R}^{\infty}_{\geq 0} \setminus \{\infty\}$
	Output:	TRUE iff $\mathcal{I}(\mathcal{K}) \leq x$
LOWER$_\mathcal{I}$	**Input**:	$\mathcal{K} \in \mathbb{K}$, $x \in \mathbb{R}^{\infty}_{\geq 0} \setminus \{0\}$
	Output:	TRUE iff $\mathcal{I}(\mathcal{K}) \geq x$

For these problems we obtain the following results.

Theorem 11. UPPER$_\mathcal{I}$ *is* NP*-complete for* $\mathcal{I} \in \{\mathcal{I}^{fuz}_{t_{prod}}, \mathcal{I}^{fuz,\Sigma}_{t_{prod}}, \mathcal{I}^{fuz}_{t_{\min}}, \mathcal{I}^{fuz,\Sigma}_{t_{\min}}\}$.

Corollary 1. *For* $\mathcal{I} \in \{\mathcal{I}^{fuz}_{t_{prod}}, \mathcal{I}^{fuz,\Sigma}_{t_{prod}}, \mathcal{I}^{fuz}_{t_{\min}}, \mathcal{I}^{fuz,\Sigma}_{t_{\min}}\}$ *the problem* LOWER$_\mathcal{I}$ *is* coNP*-complete and* EXACT$_\mathcal{I}$ *is in* $\mathsf{D}^{\mathsf{P}}_1$.

	$\textsc{Exact}_{\mathcal{I}}$	$\textsc{Upper}_{\mathcal{I}}$	$\textsc{Lower}_{\mathcal{I}}$
$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$	$\mathsf{D}^{\mathsf{P}}_1$	NP-c	coNP-c
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$	$\mathsf{D}^{\mathsf{P}}_1$	NP-c	coNP-c
$\mathcal{I}^{\text{fuz}}_{t_{\min}}$	$\mathsf{D}^{\mathsf{P}}_1$	NP-c	coNP-c
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$	$\mathsf{D}^{\mathsf{P}}_1$	NP-c	coNP-c

Table 4: Computational complexity of the considered inconsistency measures (all statements are membership statements, an additionally attached "-c" also indicates completeness for the class)

In other words, the results above state that computational problems related to our measures reside on the first level of the polynomial hierarchy and thus are not harder than a classical SAT test, which is only able to differentiate consistency from inconsistency (in particular, $\textsc{Upper}_{\mathcal{I}}$ for $\mathcal{I} \in \{\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}, \mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}, \mathcal{I}^{\text{fuz}}_{t_{\min}}, \mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}\}$ is complexity-wise equivalent to a SAT test). In fact, according to the classification hierarchy of (Thimm and Wallner, 2016), our measures belong therefore to the "easiest" class of inconsistency measures (wrt. computational complexity).

6. Revisiting the General Scheme

In the previous section we investigated properties of the specific instances $\mathcal{I}^{\text{fuz}}_t$ and $\mathcal{I}^{\text{fuz},\Sigma}_t$ of the general scheme $\mathcal{I}_{\Omega,E}$. It could be observed that the individual measures $\mathcal{I}^{\text{fuz}}_t$ and $\mathcal{I}^{\text{fuz},\Sigma}_t$ behaved similarly in some cases (e. g. wrt. computational complexity) and differently in others (e. g. wrt. the rationality postulate AI). A question that arises now is if we can establish some general properties of the general scheme $\mathcal{I}_{\Omega,E}$, i. e., whether certain properties always hold. In the unconstrained general case this is, of course, not the case.

Example 7. Let S be an arbitrary set of truth values and let Ω be an arbitrary set of S-interpretations. Define $E_1(\mathcal{K}, \omega) = 1$ for all $\mathcal{K} \in \mathbb{K}$ and $\omega \in \Omega$. Then $\mathcal{I}_{\Omega,E_1}$ is the constant function 1, which cannot be regarded a meaningful inconsistency measures as it, e. g., does not satisfy CO.

However, if we impose certain constraints on the set of interpretations Ω and the evaluation function E, we can ensure that the measure $\mathcal{I}_{\Omega,E}$ has certain characteristics. Consider the following condition.

Definition 19. Let Ω be a set of S-interpretations and E an S-evaluation function. Then E is called *supra-classical* wrt. Ω if it satisfies: $\mathcal{K} \in \mathbb{K}$ is consistent iff there is $\omega \in \Omega$ with $E(\mathcal{K}, \omega) = 0$.

In other words, E is supra-classical wrt. Ω if, for consistent $\mathcal{K}$, we can always find an interpretation that represents a classical model of $\mathcal{K}$, i. e., one which is evaluated to zero. The following observation follows immediately from this definition.

Proposition 4. *Let Ω be a set of S-interpretations and E an S-evaluation function. E is supra-classical wrt. Ω if and only if $\mathcal{I}_{\Omega,E}$ satisfies* CO.

Note that the evaluation function in Example 7 is indeed not supra-classical wrt. any set Ω.

So when developing new instances of the general scheme one should focus on supra-classical evaluation functions as only those guarantee that the induced measure satisfies the CO postulate which is the minimal requirement for any inconsistency measure. In a similar vain, we can define further conditions as follows.

Definition 20. Let Ω be a set of S-interpretations and E an S-evaluation function. Then E is called

1. *normalized* if $0 \leq E(\mathcal{K}, \omega) \leq 1$ for all $\mathcal{K} \in \mathbb{K}$, $\omega \in \Omega$,
2. *monotonic* if $E(\mathcal{K}, \omega) \leq E(\mathcal{K} \cup \{\alpha\}, \omega)$ for all $\mathcal{K} \in \mathbb{K}$, $\alpha \in \mathcal{L}$, $\omega \in \Omega$,
3. *independent* if $E(\mathcal{K}, \omega) = E(\mathcal{K} \setminus \{\alpha\}, \omega)$ for all $\mathcal{K} \in \mathbb{K}$, $\alpha \in \mathsf{Free}(\mathcal{K})$, $\omega \in \Omega$,
4. *dominant* if $E(\mathcal{K} \cup \{\alpha\}, \omega) \leq E(\mathcal{K} \cup \{\beta\}, \omega)$ for all $\mathcal{K} \in \mathbb{K}$, $\alpha, \beta \in \mathcal{L}$, $\alpha \not\models \perp$, $\alpha \models \beta$, $\omega \in \Omega$.

Proposition 5. *Let Ω be a set of S-interpretations and E an S-evaluation function.*

1. *If E is normalized then $\mathcal{I}_{\Omega,E}$ satisfies* NO.
2. *If E is monotonic then $\mathcal{I}_{\Omega,E}$ satisfies* MO.
3. *If E is independent then $\mathcal{I}_{\Omega,E}$ satisfies* IN.
4. *If E is dominant then $\mathcal{I}_{\Omega,E}$ satisfies* DO.

We leave a deeper discussion of this issue for future work and conclude this discussion with a final note regarding computational complexity.

Proposition 6. *Let Ω be a set of S-interpretations and E an S-evaluation function. If Ω is finite and $E(\mathcal{K}, \omega)$ can be evaluated in polynomial time for every $\mathcal{K} \in \mathbb{K}$ and $\omega \in \Omega$ then* $\textsc{Upper}_{\mathcal{I}_{\Omega,E}}$ *is in* NP. *If additionally E is supra-classical wrt. Ω then* $\textsc{Upper}_{\mathcal{I}_{\Omega,E}}$ *is* NP*-complete.*

7. Related Works

The works (Hunter and Konieczny, 2010; Ma et al., 2007, 2011) also present inconsistency measures based on many-valued logics, either based on Priest's three-valued logic as $\mathcal{I}_c$ or four-valued logics which add another truth value for "undefined" that can be seen as the complement of the truth value "both true and false" (B). Also these measures, basically, count the number of propositions—or first-order atoms in (Ma et al., 2007, 2011)—receiving a non-classical truth value and furthermore differ from $\mathcal{I}_c$ by an additional normalization, e. g., in the case of (Hunter and Konieczny, 2010) by dividing the inconsistency value by the number of propositions appearing in the knowledge base. Similar reductions to our general scheme from Section 3 can be given for these approaches in a straightforward manner.

Furthermore, Grant and Hunter (2016) proposed three families of inconsistency measures based on distance functions for classical (two-valued) interpretations. For example, the *Dalal distance* d_d is defined as $d_\mathrm{d}(\omega, \omega') = |\{a \in \mathsf{At} \mid \omega(a) \neq \omega'(a)\}|$ for all $\omega, \omega' \in \Omega(\mathsf{At})$. The central notion of (Grant and Hunter, 2016) is that of a k-dilation $M_d^k(\alpha)$ of a formula α, i. e., the set of interpretations that have a distance at most k from the models of ϕ, defined via

$$M_d^k(\alpha) = \{\omega \in \Omega(\mathsf{At}) \mid \exists \omega' \in \mathsf{Mod}(\alpha), d(\omega, \omega') \leq k\}$$

Define furthermore $P_d(\{\alpha_1, \ldots, \alpha_n\}) = \{(k_1, \ldots, k_n) \mid M_d^{k_1}(\alpha_1) \cap \ldots \cap M_d^{k_n}(\alpha_n) \neq \emptyset\}$. We consider the inconsistency measures $\mathcal{I}_d^\Sigma$, $\mathcal{I}_d^{\max}$, and $\mathcal{I}_d^{\mathrm{hit}}$ from (Grant and Hunter, 2016) defined via (let d be a distance function on classical interpretations)

$$\begin{aligned}
\mathcal{I}_d^\Sigma(\mathcal{K}) &= \min\{k_1 + \ldots + k_n \mid (k_1, \ldots, k_n) \in P_d(\mathcal{K})\} \\
\mathcal{I}_d^{\max}(\mathcal{K}) &= \min\{\max\{k_1, \ldots, k_n\} \mid (k_1, \ldots, k_n) \in P_d(\mathcal{K})\} \\
\mathcal{I}_d^{\mathrm{hit}}(\mathcal{K}) &= \min\{\mathrm{hit}(k_1, \ldots, k_n) \mid (k_1, \ldots, k_n) \in P_d(\mathcal{K})\}
\end{aligned}$$

where $\mathrm{hit}(k_1, \ldots, k_n) = \sum_{i=1}^n z(k_i)$ and $z(k_i) = 1$ if $k_i > 0$ and $z(k_i) = 0$ if $k_i = 0$, cf. (Grant and Hunter, 2016) for details. Although Grant and Hunter do not use many-valued logics, our scheme $\mathcal{I}_{\Omega,E}$ is general enough to also subsume these measures as well. If $X \subseteq \Omega(\mathsf{At})$ is a set of interpretations and d a distance measure on classical interpretations, we define $d(X, \omega) = \min_{\omega' \in X} d(\omega', \omega)$ (if $X = \emptyset$ we define $d_\mathrm{d}(X, \omega) = \infty$). Consider now the following $\{\mathsf{true}, \mathsf{false}\}$-evaluation functions (let $\mathcal{K} \in \mathbb{K}, \omega \in \Omega(\mathsf{At})$ and let d be a distance function on classical interpretations)

$$E_d^{\sum}(\mathcal{K},\omega) = \sum_{\alpha\in\mathcal{K}} d(\mathsf{Mod}(\alpha),\omega)$$
$$E_d^{\max}(\mathcal{K},\omega) = \max_{\alpha\in\mathcal{K}} d(\mathsf{Mod}(\alpha),\omega)$$
$$E_d^{\text{hit}}(\mathcal{K},\omega) = |\{\alpha\in\mathcal{K} \mid d(\mathsf{Mod}(\alpha),\omega) > 0\}|$$

Using these functions we can characterize the measures from (Grant and Hunter, 2016) using our scheme as follows.

Proposition 7. $\mathcal{I}_{\Omega(\mathsf{At}),E_d^{\sum}} = \mathcal{I}_d^{\Sigma}$, $\mathcal{I}_{\Omega(\mathsf{At}),E_d^{\max}} = \mathcal{I}_d^{\max}$, $\mathcal{I}_{\Omega(\mathsf{At}),E_d^{hit}} = \mathcal{I}_d^{hit}$.

8. Summary and Conclusion

We presented a general scheme for developing inconsistency measures based on many-valued logics, showed that several inconsistency measures from the literature are subsumed by this scheme and analysed its general properties. We also developed two novel families of inconsistency measures following this scheme based on fuzzy logic and parametrized by the choice of the used T-norms and T-conorms. We investigated the properties of these families by analyzing their compliance with rationality postulates, their expressivity, and their computational complexity. All measures developed in this paper have been implemented using the *Tweety libraries for Knowledge Representation and Artificial Intelligence* (Thimm, 2014) and the source code is available online.[7]

Current and future work comprises, among others, of continue investigating constraints for evaluation functions E as discussed in Section 6. Furthermore, we are investigating instantiations using other approaches to uncertain reasoning such as Dempster-Shafer theory (Shafer, 1976).

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Appendix A. Proofs of Technical Results

Proposition 1. $\mathcal{I}_{\Omega_P,E_{\min}} = \mathcal{I}_\eta$.

Proof. Let $\mathcal{K} \in \mathbb{K}$ and $\mathcal{I}_\eta(\mathcal{K}) = x$. Then there is a probability function $P \in \mathcal{P}(\mathsf{At})$ with $P(\alpha) \geq 1 - x$ for all $\alpha \in \mathcal{K}$. Let $\omega_P \in \Omega_P$ with $P(\phi) = \omega_P(\phi)$ for all $\phi \in \mathcal{L}$. Then $E_{\min}(\mathcal{K}, \omega_P) \leq 1 - x$ and therefore $\mathcal{I}_{\Omega_P,E_{\min}}(\mathcal{K}) \leq \mathcal{I}_\eta(\mathcal{K})$. The proof of $\mathcal{I}_{\Omega_P,E_{\min}}(\mathcal{K}) \geq \mathcal{I}_\eta(\mathcal{K})$ is analogous. □

Proposition 2. $\mathcal{I}_{\Omega_3,E_\#} = \mathcal{I}_c$.

Proof. Let $\mathcal{K} \in \mathbb{K}$ and $\mathcal{I}_c(\mathcal{K}) = x$. Then there is a three-valued interpretation υ with $\upsilon \models^3 \mathcal{K}$ and $x = |\upsilon^{-1}(B)|$. Let $\omega_3 \in \Omega_3$ be the corresponding interpretation with $\upsilon(\phi) = \omega_3(\phi)$ for all $\phi \in \mathcal{L}$. Then $E_\#(\omega_3) = x$ and therefore $\mathcal{I}_{\Omega_3,E_\#}(\mathcal{K}) \leq \mathcal{I}_c(\mathcal{K})$. The proof of $\mathcal{I}_{\Omega_3,E_\#}(\mathcal{K}) \geq \mathcal{I}_c(\mathcal{K})$ is analogous. □

Proposition 3.

$$\mathcal{I}^{\mathit{fuz}}_{t_{\min}}(\mathcal{K}) = \begin{cases} 1/2 & \text{\textit{if} } \mathcal{K} \models \perp \\ 0 & \text{\textit{otherwise}} \end{cases}$$

Proof. Let $\mathcal{K}$ be consistent. Then there is $\omega \in \Omega(\mathsf{At})$ with $\omega \models \mathcal{K}$. Consider the fuzzy interpretation $\omega' \in \Omega_{t_{\min}}$ defined via $\omega'(a) = 1$ for all $a \in \mathsf{At}$ with $\omega(a) = \mathsf{true}$, $\omega'(a) = 0$ for all $a \in \mathsf{At}$ with $\omega(a) = \mathsf{false}$, and $\omega'(\alpha)$ defined as prescribed by Definition 11 for all other $\alpha \in \mathcal{L}$. Then $\omega'(\phi) = 1$ for all $\phi \in \mathcal{K}$ and $E^{\mathrm{fuz}}_{t_{\min}}(\mathcal{K}, \omega') = 0$. Then $\mathcal{I}^{\mathrm{fuz}}_{t_{\min}}(\mathcal{K}) = 0$.

Assume now that $\mathcal{K}$ is inconsistent. Define $\omega' \in \Omega_{t_{\min}}$ via $\omega'(\alpha) = 1/2$ for all $\alpha \in \mathcal{L}$ (observe that ω' is indeed a fuzzy interpretation according to Definition 11 for $t = \min$ and $s = \max$). Then $E^{\mathrm{fuz}}_{t_{\min}}(\mathcal{K}, \omega') = 1/2$ and $\mathcal{I}^{\mathrm{fuz}}_{t_{\min}}(\mathcal{K}) \leq 1/2$. Assume that $\mathcal{I}^{\mathrm{fuz}}_{t_{\min}}(\mathcal{K}) = x < 1/2$. Then there is $\omega'' \in \Omega_{t_{\min}}$ with $E^{\mathrm{fuz}}_{t_{\min}}(\mathcal{K}, \omega'') = x$. Then we have

$$\begin{aligned} E^{\mathrm{fuz}}_{t_{\min}}(\mathcal{K}, \omega'') = x &= n(t_{\min}(\{\omega''(\alpha) \mid \alpha \in \mathcal{K}\})) \\ &= 1 - \min\{\omega''(\alpha) \mid \alpha \in \mathcal{K}\} \end{aligned}$$

Therefore, for all $\alpha \in \mathcal{K}$ we have $\omega''(\alpha) \geq 1 - x > 1/2$. Define now $\omega \in \Omega(\mathsf{At})$ via $\omega(a) = \mathsf{true}$ for all $a \in \mathsf{At}$ with $\omega''(a) > 1/2$ and $\omega(a) = \mathsf{false}$ for all $a \in \mathsf{At}$ with $\omega''(a) \leq 1/2$. We show now by structural induction on arbitrary $\alpha \in \mathcal{L}$ that $\omega''(\alpha) > 1/2$ if and only if $\omega \models \alpha$:

- $\alpha = a \in \mathsf{At}$: $\omega''(\alpha) > 1/2$ if and only if $\omega(\alpha) = \mathsf{true}$ (equivalent to $\omega \models \alpha$) by definition.

- $\alpha = \alpha_1 \wedge \alpha_2$: If $\omega''(\alpha) > 1/2$ then $\omega''(\alpha_1) > 1/2$ and $\omega''(\alpha_2) > 1/2$ as $\omega''(\alpha) = \min\{\omega''(\alpha_1), \omega''(\alpha_2)\}$. By inductive assumption it follows $\omega \models \alpha_1$ and $\omega \models \alpha_2$, therefore $\omega \models \alpha$. The other direction is analogous.

- $\alpha = \alpha_1 \vee \alpha_2$: If $\omega''(\alpha) > 1/2$ then $\omega''(\alpha_i) > 1/2$ for some $i \in \{1, 2\}$ as $\omega''(\alpha) = \max\{\omega''(\alpha_1), \omega''(\alpha_2)\}$. By inductive assumption it follows $\omega \models \alpha_i$, therefore $\omega \models \alpha$. The other direction is analogous.

- $\alpha = \neg\alpha_1$: If $\omega''(\alpha) > 1/2$ then $\omega''(\alpha_1) < 1/2$ as $\omega''(\alpha) = 1 - \omega''(\alpha_1)$. By inductive assumption it follows $\omega \not\models \alpha_1$, therefore $\omega \models \alpha$. The other direction is analogous.

Therefore $\omega \models \mathcal{K}$ and $\mathcal{K}$ is consistent, contrary to the assumption. Hence, $\mathcal{I}^{\mathrm{fuz}}_{t_{\min}}(\mathcal{K}) \geq 1/2$ and therefore $\mathcal{I}^{\mathrm{fuz}}_{t_{\min}}(\mathcal{K}) = 1/2$. □

Theorem 2. *Let t be any T-norm and s any T-conorm. $\mathcal{I}^{fuz}_{t,s}$ satisfies* CO*,* NO*,* MO*,* SI*,* FD*, and* AI*. In general, $\mathcal{I}^{fuz}_{t,s}$ does not satisfy* IN*,* DO*,* SA*,* PY*,* MI*,* MN*,* AT*,* EC*,* AC*,* CD*,* SY*, and* EX*.*

Proof. In the following, we denote by +X a proof that shows that property X is satisfied and by −X a proof that shows that property X is violated.

+**CO** Assume $\mathcal{K}$ is consistent. There there is $\omega \in \Omega(\mathsf{At})$ with $\omega \models \mathcal{K}$. Define a fuzzy interpretation function ω_F via

1. for all $a \in \mathsf{At}$, $\omega_F(a) = 1$ iff $\omega(a) = \mathsf{true}$ and $\omega_F(a) = 0$ iff $\omega(a) = \mathsf{false}$
2. $\omega_F(\neg\alpha) = n(\omega_F(\alpha))$
3. $\omega_F(\alpha \wedge \beta) = t(\omega_F(\alpha), \omega_F(\beta))$
4. $\omega_F(\alpha \vee \beta) = s(\omega_F(\alpha), \omega_F(\beta))$

Note that ω_F is indeed a fuzzy interpretation function according to Definition 11. As t is a T-norm according to Definition 9 and s is a T-conorm according to Definition 10, due to $\omega \models \alpha$ for all $\alpha \in \mathcal{K}$ we have $\omega_F(\alpha) = 1$ for all $\alpha \in \mathcal{K}$ and (due to neutrality of t, $E_t^{\text{fuz}}(\mathcal{K}, \omega_F) = 0$. It follows $\mathcal{I}_{t,s}^{\text{fuz}}(\mathcal{K}) = 0$.

Assume now $\mathcal{I}_{t,s}^{\text{fuz}}(\mathcal{K}) = 0$ and let $\mathcal{K} = \{\alpha_1, \ldots, \alpha_n\}$. Then there is $\omega_F \in \Omega_{t,s}$ with $t(\{\omega(\alpha_1), \ldots, \omega(\alpha_n)\} = 1$. By *neutrality* of t it follows $\omega(\alpha_1) = \ldots = \omega(\alpha_n) = 1$. Due to properties 4 and 5 of Definitions 9 and 10, respectively, it follows $\omega_F(a) \in \{0, 1\}$ for all $a \in \mathsf{At}$. In analogy to above, we can construct $\omega \in \Omega(\mathsf{At})$ with $\omega \models \mathcal{K}$. Therefore, $\mathcal{K}$ is consistent.

+**NO** It is $\mathcal{I}_{t,s}^{\text{fuz}}(\mathcal{K}) = E_t^{\text{fuz}}(\mathcal{K}, \omega) = n(t(\{\omega(\alpha) \mid \alpha \in \mathcal{K}\}))$ for some $\omega \in \Omega_{t,s}$. As the range of t is $[0, 1]$ by definition, the claim follows.

+**MO** Note that for $S, S' \subseteq [0, 1]$ with $S \subseteq S'$ we have $t(S) \geq t(S')$ for any T-norm t. Therefore $E_t^{\text{fuz}}(\mathcal{K}, \omega) \leq E_t^{\text{fuz}}(\mathcal{K}', \omega)$ for every $\omega \in \Omega_{t,s}$ and $\mathcal{K} \subseteq \mathcal{K}'$ and therefore $\mathcal{I}_{t,s}^{\text{fuz}}(\mathcal{K}) \leq \mathcal{I}_{t,s}^{\text{fuz}}(\mathcal{K}')$.

+**SI** Let $\alpha \in \mathsf{Safe}(\mathcal{K})$. As $\alpha \not\models \perp$ there is $\omega \in \Omega(\mathsf{At})$ with $\omega \models \alpha$. Similarly to the proof of **CO**, let ω_F be a fuzzy interpretation on the propositions of α such that $\omega_F(\alpha) = 1$. Let ω'_F be a fuzzy interpretation on the propositions of $\mathcal{K} \setminus \{\alpha\}$ such that $\mathcal{I}_{t,s}^{\text{fuz}}(\mathcal{K} \setminus \{\alpha\}) = E_t^{\text{fuz}}(\mathcal{K} \setminus \{\alpha\}, \omega'_F)$. As the domains of ω_F and ω'_F are disjoint, define ω''_F via $\omega''_F(a) = \omega_F(a)$ for all propositions in α, $\omega''_F(a) = \omega'_F(a)$ for all propositions in $\mathcal{K} \setminus \{\alpha\}$. Then

$$\begin{aligned}\mathcal{I}^{\text{fuz}}_{t,s}(\mathcal{K}) &\leq E^{\text{fuz}}_t(\mathcal{K}, \omega''_F)\\ &= n(t(\{\omega''_F(\phi) \mid \phi \in \mathcal{K} \setminus \{\alpha\}\} \cup \{\omega''_F(\alpha)\}))\\ &= n(t(\{\omega''_F(\phi) \mid \phi \in \mathcal{K} \setminus \{\alpha\}\} \cup \{1\}))\\ &= n(t(\{\omega''_F(\phi) \mid \phi \in \mathcal{K} \setminus \{\alpha\}\}))\\ &= \mathcal{I}^{\text{fuz}}_{t,s}(\mathcal{K} \setminus \{\alpha\})\end{aligned}$$

$\mathcal{I}^{\text{fuz}}_{t,s}(\mathcal{K}) \geq \mathcal{I}^{\text{fuz}}_{t,s}(\mathcal{K} \setminus \{\alpha\})$ follows from MO.

$+$**FD** This follows directly from MO.

$+$**AI** This follows directly from the associativity of t and $\omega(\alpha \wedge \beta) = t(\omega(\alpha), \omega(\beta)) = t(\{\omega(\alpha), \omega(\beta)\})$.

$-$**IN** Consider $\mathcal{K}^1 = \{a, \neg a\}$ and observe $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^1) = 0.75$, but $a \vee \neg a \in \mathsf{Free}(\mathcal{K}^1 \cup \{a \vee \neg a\})$ and $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^1 \cup \{a \vee \neg a\}) = 0.8125$.

$-$**DO** Consider $\mathcal{K}^1 = \{a, \neg a\}$ and observe $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^1) = 0.75$, but $a \not\models \perp$, $a \models a \wedge a$, and $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\{a \wedge a, \neg a\}) = 23/27 > 0.75$ (the value is derived for $\omega_F(a) = 2/3$).

$-$**SA** Consider $\mathcal{K}^1 = \{a, \neg a\}$ and observe $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^1) = 0.75$. Analogously, $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\{b, \neg b\}) = 0.75$. By SA it would follow $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^1 \cup \{b, \neg b\}) \geq 1.5$, violating NO.

$-$**PY** Consider $\mathcal{K}^1 = \{a, \neg a\}$ and observe $\mathcal{I}^{\text{fuz}}_{t_{\min}}(\mathcal{K}^1) = 0.5$ and $\mathcal{I}^{\text{fuz}}_{t_{\min}}(\mathcal{K}^1 \cup \{a \wedge a\}) = 0.5$, although $a \wedge a \notin \mathsf{Free}(\mathcal{K}^1 \cup \{a \wedge a\})$.

$-$**MI** See the counterexample of SA.

$-$**MN** Consider $\mathcal{K}^1 = \{a, \neg a\}$ and observe $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^1) = 0.75$, but $\mathcal{K}^1 \in \mathsf{MI}(\mathcal{K}^1)$.

$-$**AT** Consider $\mathcal{K}^1 = \{a, \neg a\}$, $\mathcal{K}^2 = \{a \wedge a, \neg a \wedge \neg a\}$ (both minimally inconsistent) and observe $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^1) = 0.75$, $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^2) = 0.9375$, but $|\mathcal{K}^1| = |\mathcal{K}^2|$.

$-$**EC** Consider $\mathcal{K}^1 = \{a, \neg a\}$, $\mathcal{K}^3 = \{a \wedge \neg a\}$ (both minimally inconsistent) and observe $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^1) = \mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^3) = 0.75$, but $|\mathcal{K}^1| > |\mathcal{K}^3|$.

$-$**AC** Consider $\mathcal{K}^{4,i} = \{a_1, \ldots, a_i, \neg a_1 \vee \ldots \vee \neg a_i\}$ for $i \in \mathbb{N}$ and observe $\lim_{i\to\infty} |\mathcal{K}_i^{4,i}| = \infty$ and each $\mathcal{K}^{4,i}$ is minimally inconsistent. However, $\mathcal{I}^{\text{fuz}}_{t_{\min}}(\mathcal{K}^{4,i}) = 0.5$ for all $i \in \mathbb{N}$.

$-$**CD** Consider $\mathcal{K}^3 = \{a \wedge \neg a\}$ and observe that every non-empty subset of $\mathcal{K}^3$ is inconsistent, but $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^3) = 0.75$.

$-$**SY** Due to $a \equiv a \wedge a$ the same counterexample as for DO can be used.

$-$**EX** Due to $a \equiv a \wedge a$ the same counterexample as for DO can be used. □

Theorem 3. *Let t be any T-norm and s any T-conorm. $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$ satisfies CO, MO, SI, SA, and FD. $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$ does not satisfy NO, IN, DO, PY, MI, MN, AT, EC, AC, CD, SY, EX, and AI.*

Proof. In the following, we denote by +X a proof that shows that property X is satisfied and by $-$X a proof that shows that property X is violated.

+**CO** Analogous to the corresponding proof for $\mathcal{I}^{\text{fuz}}_{t,s}$ (see Theorem 2). Additionally observe, if $\omega_F(\alpha) = 1$ for all $\alpha \in \mathcal{K}$ it follows directly $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}(\mathcal{K}) = 0$; and if $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}(\mathcal{K}) = 0$ it also implies $\omega(\alpha) = 1$ for some $\omega \in \Omega_{t,s}$ and all $\alpha \in \mathcal{K}$.

+**MO** Let $\mathcal{K} \subseteq \mathcal{K}'$ and observe

$$\begin{aligned} E^{\text{fuz}}_{\Sigma}(\mathcal{K}, \omega) &= \sum_{\alpha \in \mathcal{K}} n(\omega(\alpha)) \\ &\leq \sum_{\alpha \in \mathcal{K}} n(\omega(\alpha)) + \sum_{\alpha \in \mathcal{K}' \setminus \mathcal{K}} n(\omega(\alpha)) \\ &= E^{\text{fuz}}_{\Sigma}(\mathcal{K}', \omega) \end{aligned}$$

for every $\omega \in \Omega_{s,t}$. It follows $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}(\mathcal{K}) \leq \mathcal{I}^{\text{fuz},\Sigma}_{t,s}(\mathcal{K}')$.

+**SI** Analogous to the corresponding proof for $\mathcal{I}^{\text{fuz}}_{t,s}$ (see Theorem 2).

+**SA** Let $\mathcal{K}_1, \mathcal{K}_2$, $\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset$, and $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. Let $\omega \in \Omega_{t,s}$ such that

$\mathcal{I}^{\text{fuz},\Sigma}_{t,s}(\mathcal{K}) = E^{\text{fuz}}_{\Sigma}(\mathcal{K},\omega)$. Then

$$\begin{aligned}\mathcal{I}^{\text{fuz},\Sigma}_{t,s}(\mathcal{K}) &= E^{\text{fuz}}_{\Sigma}(\mathcal{K},\omega)\\ &= \sum_{\alpha\in\mathcal{K}} n(\omega(\alpha))\\ &= \sum_{\alpha\in\mathcal{K}_1} n(\omega(\alpha)) + \sum_{\alpha\in\mathcal{K}_2} n(\omega(\alpha))\\ &= E^{\text{fuz}}_{\Sigma}(\mathcal{K}_1,\omega) + E^{\text{fuz}}_{\Sigma}(\mathcal{K}_2,\omega)\\ &\geq \mathcal{I}^{\text{fuz},\Sigma}_{t,s}(\mathcal{K}_1) + \mathcal{I}^{\text{fuz},\Sigma}_{t,s}(\mathcal{K}_2)\end{aligned}$$

$+$**FD** This follows directly from MO.

$-$**NO** Consider $\mathcal{K}^5 = \{a, \neg a, b, \neg b\}$ and observe $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}^5) = 2$.

$-$**IN** Consider

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\{a\wedge\neg a, a\wedge a\wedge\neg a\wedge\neg a\}) = 1.6875$$
$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\{a\wedge\neg a, a\wedge a\wedge\neg a\wedge\neg a, a\}) = 2$$

and note $a \in \mathsf{Free}(\{a\wedge\neg a, a\wedge a\wedge\neg a\wedge\neg a, a\})$.

$-$**DO** Consider $\mathcal{K}^7$ defined via

$$\begin{aligned}\mathcal{K}^7 = \{&a\wedge\neg a, \neg\neg(a\wedge\neg a), \neg\neg\neg\neg(a\wedge\neg a),\\ &\neg\neg\neg\neg\neg\neg\neg(a\wedge\neg a), \neg\neg\neg\neg\neg\neg\neg\neg\neg\neg(a\wedge\neg a)\}\end{aligned}$$

and observe

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^7\cup\{a\}) = 4.2$$

Observe now $a \not\models \perp$, $a \models a\wedge a$, and

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^7\cup\{a\wedge a\}) = 4.4375$$

violating DO.

$-$**PY** Consider $\mathcal{K}^1 = \{a, \neg a\}$ and observe $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^1) = 1$ and $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^1\cup\{a\wedge a\}) = 1$, although $a\wedge a \notin \mathsf{Free}(\mathcal{K}^1\cup\{a\wedge a\})$.

$-$**MI** $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$ cannot satisfy MI as this would imply IN.

–**MN** Consider $\mathcal{K}^3 = \{a \wedge \neg a\}$ and observe $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^3) = 0.75$, but $\mathcal{K}^3 \in \mathsf{MI}(\mathcal{K}^3)$.

–**AT** Consider $\mathcal{K}^3 = \{a \wedge \neg a\}$ with $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^3) = 0.75$ and $\mathcal{K}^1 = \{a, \neg a\}$ with $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^1) = 1$.

–**EC** Consider $\mathcal{K}^1 = \{a, \neg a\}$ and observe $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^1) = 1$. Furthermore, we have for $\mathcal{K}^6 = \{a, b, \neg a \vee \neg b\}$, $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^6) = 1$ as well, but $|\mathcal{K}^1| < |\mathcal{K}^6|$.

–**AC** Consider $\mathcal{K}^{4,i} = \{a_1, \ldots, a_i, \neg a_1 \vee \ldots \vee \neg a_i\}$ for $i \in \mathbb{N}$ and observe $\lim_{i\to\infty} |\mathcal{K}^{4,i}_i| = \infty$ and each $\mathcal{K}^{4,i}$ is minimally inconsistent. However, $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^{4,i}) = 1$ for all $i \in \mathbb{N}$.

–**CD** Consider $\mathcal{K}^3 = \{a \wedge \neg a\}$ and observe that every non-empty subset of $\mathcal{K}^3$ is inconsistent, but $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}^3) = 0.75$.

–**SY** Consider

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\{a \wedge \neg a, b \wedge \neg b\}) = 1.5$$
$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\{a \wedge \neg a, b \wedge \neg b \wedge a \wedge \neg a\}) = 1.6875$$

and note $b \wedge \neg b \equiv b \wedge \neg b \wedge a \wedge \neg a$.

–**EX** Consider

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\{a, \neg a, b, \neg b\}) = 2$$
$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\{a, \neg a \wedge b, \neg b\}) = 1$$

but $\{\neg a, b\} \not\models \perp$ and $\{\neg a, b\} \equiv \{\neg a \wedge b\}$.

–**AI** Consider $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\{a, \neg a\}) = 1$ but $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\{a \wedge \neg a\}) = 0.75$. □

Theorem 5. *$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$ satisfies CO, NO, MO, SI, FD, and AI. $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$ does not satisfy IN, DO, SA, PY, MI, MN, AT, EC, AC, CD, SY, and EX.*

Proof. As $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$ is an instance of $\mathcal{I}^{\text{fuz}}_{t,s}$, satisfaction of CO, NO, MO, SI, FD, and AI follows directly from Theorem 2. Violation of IN, DO, SA, MI, MN, AT, EC, CD, SY, and EX is also due to the proof of Theorem 2 as there $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$ was used as a counterexample. It remains to show that $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$ does not satisfy PY and AC.

−**PY** Consider $\mathcal{K}^{8,2} = \{a_1, a_2, \neg(a_1 \wedge a_2)\}$ and $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^{8,2}) = 0.75$ (using ω with $\omega(a_1) = 1$ and $\omega(a_2) = 0.5$, see above). Then $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^{8,2} \cup \{a_1 \wedge a_1\}) = 0.75$ as well (using the very same ω as before), violating PY as $a_1 \wedge a_1$ is not free in $\mathcal{K}^{8,2} \cup \{a_1 \wedge a_1\}$.

−**AC** Consider now $\mathcal{K}^{8,i} = \{a_1, \ldots, a_i, \neg(a_1 \wedge \ldots \wedge a_i)\}$ for $i \in \mathbb{N}$ and observe $\lim_{i\to\infty} |\mathcal{K}^{8,i}_i| = \infty$ and each $\mathcal{K}^{4,i}$ is minimally inconsistent. Now $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^{8,i})$ can be written as

$$\begin{aligned}
\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^{8,i}) &= \min_{x_1,\ldots,x_i \in [0,1]} (1 - (x_1 \cdot \ldots \cdot x_i(1 - x_1 \cdot \ldots \cdot x_i))) \\
&= 1 - \max_{x_1,\ldots,x_i \in [0,1]} (\underbrace{x_1 \cdot \ldots \cdot x_i}_{X}(1 - \underbrace{x_1 \cdot \ldots \cdot x_i}_{X})) \\
&= 1 - \max_{X \in [0,1]} X(1 - X) \\
&= 1 - 0.5(1 - 0.5) = 0.75
\end{aligned}$$

with $0.5 = X = x_1 \cdot \ldots \cdot x_i$, e. g., $x_1 = \ldots = x_{i-1} = 1$ and $x_i = 0.5$. This shows that $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}^{8,i})$ violates AC. □

Theorem 6. *$\mathcal{I}^{fuz,\Sigma}_{t_{\min}}$ satisfies CO, MO, DO, SI, SA, FD and SY. $\mathcal{I}^{fuz,\Sigma}_{t_{\min}}$ does not satisfy NO, IN, PY, MI, MN, AT, EC, AC, CD, EX, and AI.*

Proof. As $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$ is an instance of $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$, satisfaction of CO, MO, SI, SA, and FD follows directly from Theorem 3. Violation of NO is also due to the proof of Theorem 3 as there $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$ was used as a counterexample.

In the following, we denote by +X a proof that shows that property X is satisfied and by −X a proof that shows that property X is violated.

+**DO** Let α, β be formulas with $\alpha \not\models \perp$ and $\alpha \models \beta$. Then for any $\omega \in \Omega_{t_{\min}}$ we have $\omega(\alpha) \leq \omega(\beta)$, cf. (Hájek, 1998). Let now $\mathcal{K}$ be any knowledge base and ω such that

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K} \cup \{\alpha\}) = E^{\text{fuz}}_{\Sigma}(\mathcal{K} \cup \{\alpha\}, \omega)$$

Then (recall that $n(x) = 1 - x$)

$$\begin{aligned} E^{\text{fuz}}_{\Sigma}(\mathcal{K} \cup \{\alpha\}, \omega) &= \sum_{\gamma \in \mathcal{K} \cup \{\alpha\}} n(\omega(\gamma)) \\ &= \sum_{\gamma \in \mathcal{K}} n(\omega(\gamma)) + n(\omega(\alpha)) \\ &\geq \sum_{\gamma \in \mathcal{K}} n(\omega(\gamma)) + n(\omega(\beta)) \\ &= E^{\text{fuz}}_{\Sigma}(\mathcal{K} \cup \{\beta\}, \omega) \\ &\geq \mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K} \cup \{\beta\}) \end{aligned}$$

showing DO.

– **IN** Consider

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\{a \wedge \neg a, a \wedge a \wedge \neg a \wedge \neg a\}) = 1$$
$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\{a \wedge \neg a, a \wedge a \wedge \neg a \wedge \neg a, a\}) = 1.5$$

and note $a \in \mathsf{Free}(\{a \wedge \neg a, a \wedge a \wedge \neg a \wedge \neg a, a\})$.

– **PY** Consider $\mathcal{K}^1 = \{a, \neg a\}$ and observe $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}^1) = 1$ and $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}^1 \cup \{a \wedge a\}) = 1$, although $a \wedge a \notin \mathsf{Free}(\mathcal{K}^1 \cup \{a \wedge a\})$.

– **MI** $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$ cannot satisfy MI as this would imply IN.

– **MN** Consider $\mathcal{K}^3 = \{a \wedge \neg a\}$ and observe $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}^3) = 0.5$, but $\mathcal{K}^3 \in \mathsf{MI}(\mathcal{K}^3)$.

– **AT** Consider $\mathcal{K}^3 = \{a \wedge \neg a\}$ with $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}^3) = 0.5$ and $\mathcal{K}^1 = \{a, \neg a\}$ with $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}^1) = 1$.

– **EC** Consider $\mathcal{K}^1 = \{a, \neg a\}$ and observe $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}^1) = 1$. Furthermore, we have for $\mathcal{K}^6 = \{a, b, \neg a \vee \neg b\}$, $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}^6) = 1$ as well, but $|\mathcal{K}^1| < |\mathcal{K}^6|$.

– **AC** Consider $\mathcal{K}^{4,i} = \{a_1, \ldots, a_i, \neg a_1 \vee \ldots \vee \neg a_i\}$ for $i \in \mathbb{N}$ and observe $\lim_{i\to\infty} |\mathcal{K}^{4,i}_i| = \infty$ and each $\mathcal{K}^{4,i}$ is minimally inconsistent. However, $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}^{4,i}) = 1$ for all $i \in \mathbb{N}$.

– **CD** Consider $\mathcal{K}^3 = \{a \wedge \neg a\}$ and observe that every non-empty subset of $\mathcal{K}^3$ is inconsistent, but $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}^3) = 0.5$.

– **SY** Let $\mathcal{K}_1 = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathcal{K}_2 = \{\beta_1, \ldots, \beta_n\}$ with $\alpha_i \equiv \beta_i$ for all $i = 1, \ldots, n$. It follows $\mathcal{K}_1 \equiv^s \mathcal{K}_2$. For any $\omega \in \Omega_{t_{\min}}$, we have $\omega(\alpha_i) = \omega(\beta_i)$, cf. (Hájek, 1998). The claim follows directly.

– **EX** Consider

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\{a, \neg a, b, \neg b\}) = 2$$
$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\{a, \neg a \wedge b, \neg b\}) = 1$$

but $\{\neg a, b\} \not\models \perp$ and $\{\neg a, b\} \equiv \{\neg a \wedge b\}$.

– **AI** Consider $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\{a, \neg a\}) = 1$ but $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\{a \wedge \neg a\}) = 0.5$. □

Theorem 7. *$\mathcal{I}^{fuz,\Sigma}_{t_{prod}}$ satisfies CO, MO, SI, SA, and FD. $\mathcal{I}^{fuz,\Sigma}_{t_{prod}}$ does not satisfy NO, IN, DO, PY, MI, MN, AT, EC, AC, CD, SY, EX, and AI.*

Proof. As $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$ is an instance of $\mathcal{I}^{\text{fuz},\Sigma}_{t,s}$, satisfaction of CO, MO, SI, SA, and FD follows directly from Theorem 3. Violation of the other postulates (except NO) is also due to the proof of Theorem 3 as there $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$ was used as a counterexample. $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$ also violates NO as, e. g., $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\{a, \neg a, b, \neg b\}) = 2$. □

Theorem 8. *For all $n > 0$, $\mathcal{C}^v(\mathcal{I}^{fuz}_{t_{prod}}, n) = \mathcal{C}^f(\mathcal{I}^{fuz}_{t_{prod}}, n) = \mathcal{C}^p(\mathcal{I}^{fuz}_{t_{prod}}, n) = \infty$. For all $n > 1$, $\mathcal{C}^l(\mathcal{I}^{fuz}_{t_{prod}}, n) = \infty$.*

Proof. Consider the family $\hat{\mathcal{K}}^{1,i}$ of knowledge bases defined via

$$\hat{\mathcal{K}}^{1,i} = \{\neg a \wedge \underbrace{a \wedge \ldots \wedge a}_{i \text{ times}}\}$$

for $i \in \mathbb{N}$. Then for every $\omega \in \Omega_{t_{\text{prod}}}$

$$E^{\text{fuz}}_{t_{\text{prod}}}(\hat{\mathcal{K}}^{1,i}, \omega) = n(t_{\text{prod}}(\{\omega(\neg \alpha), \underbrace{\omega(\alpha), \ldots, \omega(\alpha)}_{i \text{ times}}\}))$$
$$= n(n(\omega(\alpha)))\omega(\alpha)^i$$

and therefore

$$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\hat{\mathcal{K}}^{1,i}) = \min_{x \in [0,1]} n(n(x))x^i$$
$$= 1 - \max_{x \in [0,1]} n(x)x^i$$

Note that the function $f(x) = n(x)x^i$ is maximal for $x = i/(i+1)$ and therefore $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\hat{\mathcal{K}}^{1,i}) = n(i/(i+1)) = 1/(i+1)$. As $\mathsf{At}(\hat{\mathcal{K}}^{1,i}) = 1$, $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$ and $|\hat{\mathcal{K}}^{1,i}| = 1$ can attain infinitely many values for knowledge bases with at least 1 proposition and 1 formula, therefore $\mathcal{C}^v(\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}, n) = \infty$ and $\mathcal{C}^f(\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}, n) = \infty$.

Consider now $\hat{\mathcal{K}}^{2,i}$ defined via

$$\hat{\mathcal{K}}^{2,i} = \{a_1, \ldots, a_i, \neg a_1, \ldots, \neg a_i\}$$

Then $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\hat{\mathcal{K}}^{2,i}) = n(1/2^{2i})$ showing $\mathcal{C}^l(\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}, n) = \mathcal{C}^p(\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}, n) = \infty$. □

Theorem 9. *For all* $n > 0$, $\mathcal{C}^v(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}, n) = \mathcal{C}^f(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}, n) = \mathcal{C}^p(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}, n) = \infty$. *For all* $n > 1$, $\mathcal{C}^l(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}, n) = \infty$.

Proof. The proof is analogous to the proof of Theorem 8. Indeed, note that $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}) = \mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K})$ for every $\mathcal{K}$ with $|\mathcal{K}| = 1$, so $\hat{\mathcal{K}}^{1,i}$ can also be used to show $\mathcal{C}^v(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}, n) = \mathcal{C}^f(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}, n) = \infty$. Observe also $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\hat{\mathcal{K}}^{2,i}) = i$ showing $\mathcal{C}^l(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}, n) = \mathcal{C}^p(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}, n) = \infty$ □

Theorem 10. *For all* $n > 0$, $\mathcal{C}^v(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, n) = \infty$, $\mathcal{C}^f(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, n) = n+1$, *and* $\mathcal{C}^p(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, n) = \infty$. *For all* $n > 1$, $\mathcal{C}^l(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, n) = \infty$.

Proof. Consider the family $\hat{\mathcal{K}}^{3,i}$ of knowledge bases defined via

$$\hat{\mathcal{K}}^{3,i} = \{\neg a, a, a \wedge a, \neg a \wedge \neg a, \ldots, \underbrace{a \wedge \ldots \wedge a}_{i \text{ times}}, \underbrace{\neg a \wedge \ldots \wedge \neg a}_{i \text{ times}}\}$$

for $i \in \mathbb{N}$. Then for every $\omega \in \Omega_{t_{\min}}$ we can simply write (note that e. g. $\omega(a) = \omega(a \wedge a)$ for $\omega \in \Omega_{t_{\min}}$)

$$\begin{aligned} E^{\text{fuz}}_{\Sigma}(\hat{\mathcal{K}}^{1,i}, \omega) &= \sum_{j=1}^{i} (n(\omega(a)) + n(\omega(\neg a))) \\ &= \sum_{j=1}^{i} ((1 - \omega(a)) + (1 - (1 - \omega(a))) \\ &= i \end{aligned}$$

and therefore $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\hat{\mathcal{K}}^{3,i}) = i$, showing $\mathcal{C}^v(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, n) = \infty$.

As for $\mathcal{C}^f(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, n)$, note first that $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}) \leq n/2$ for any $\mathcal{K}$ with $|\mathcal{K}| \leq n$ as $\hat{\omega} \in \Omega_{t_{\min}}$ defined via $\omega(a) = 1/2$ for all $a \in \mathsf{At}$ yields $\omega(\alpha) \geq 1/2$ for all $\alpha \in \mathcal{K}$ and thus $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}) \leq n/2$. More concretely, we claim that the range of $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$ on knowledge bases with at most n formulas is the set $R_n = \{0, 1/2, 1, 3/2, \ldots, n/2\}$. To see this, consider the family $\hat{\mathcal{K}}^{4,i}$ of knowledge bases defined via

$$\hat{\mathcal{K}}^{4,i} = \{\neg a_1 \wedge a_1, \ldots, \neg a_i \wedge a_i\}$$

for $i \in \mathbb{N}$ with $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\hat{\mathcal{K}}^{4,i}) = i/2$, effectively showing that the range is at least a superset of R_n (note that for $i = 0$, $\hat{\mathcal{K}}^{4,i} = \emptyset$ with $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\hat{\mathcal{K}}^{4,i}) = 0$). To show that R_n is indeed exactly the range, we can use a similar argumentation line as in the proof of Proposition 3 (which showed that $\mathcal{I}^{\text{fuz}}_{t_{\min}}$ only attains the values 0 and $1/2$). In particular, if $\omega_0 \in \Omega_{t_{\min}}$ is such that $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}) = E^{\text{fuz}}_{\Sigma}(\mathcal{K}, \omega)$ then it follows $\omega(\alpha) \in \{0, 1/2, 1\}$ for all $\alpha \in \mathcal{K}$. Together with the observation $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}) \leq n/2$ from above, it follows that R_n is indeed the range of $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$ on knowledge bases with at most n formulas and thus $\mathcal{C}^f(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, n) = n + 1$.

As for $\mathcal{C}^p(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, n)$, note that $\hat{\mathcal{K}}^{4,i}$ mentions only one proposition in each formula and due to $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\hat{\mathcal{K}}^{4,i}) = i/2$ it follows that $\mathcal{C}^p(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, n) = \infty$ for $n > 0$.

Consider finally the family $\hat{\mathcal{K}}^{5,i}$ of knowledge bases defined via

$$\hat{\mathcal{K}}^{5,i} = \{\neg a_1, a_1, \ldots, \neg a_i, a_i\}$$

for $i \in \mathbb{N}$ and note that all formulas have length at most 2. As $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\hat{\mathcal{K}}^{5,i}) = i$ it follows $\mathcal{C}^l(\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, n) = \infty$ for $n > 1$. □

Theorem 11. UPPER$_{\mathcal{I}}$ *is* NP*-complete for* $\mathcal{I} \in \{\mathcal{I}^{fuz}_{t_{prod}}, \mathcal{I}^{fuz,\Sigma}_{t_{prod}}, \mathcal{I}^{fuz}_{t_{\min}}, \mathcal{I}^{fuz,\Sigma}_{t_{\min}}\}$.

Proof. We only consider $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$, the proofs for the other measures are analogous. We first show NP-membership. Let $(\mathcal{K}, x)$ be an instance of UPPER$_{\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}}$ and let $\{a_1, \ldots, a_n\}$ be the propositions appearing in $\mathcal{K}$. In order to continue we need the following observation.

Lemma 1. *If* $\mathcal{I}^{fuz}_{t_{prod}}(\mathcal{K}) \leq x$ *then there is* $\omega \in \Omega_{t_{prod}}$ *with* $E^{fuz}_{t_{prod}}(\mathcal{K}, \omega) \leq x$ *such that the length of the binary encoding of each* $\omega(a_i)$ *is smaller or equal to the length of the binary encoding of* x*.*

The proof of Lemma 1 follows from simple arithmetics. Note that $E^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K},\omega) \in [0,1]$ and that $E^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K},\omega)$ is a function on $\omega(a_1),\ldots,\omega(a_n) \in [0,1]$ and composed of functions $t_{\text{prod}}(x,y) = xy$ and $n(x) = 1-x$. Observe that the length of the binary encoding of $n(x)$ is the same as for x and that the length of the binary encoding of $t_{\text{prod}}(x,y)$ is at least as long as the maximum length of the encodings of x and y.

We return to the proof of Theorem 11 and define a non-deterministic algorithm for $\textsc{Upper}_{\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}}$. First, non-deterministically guess $y_1,\ldots,y_n \in [0,1]$ such that the length of the binary encoding of each y_i is smaller or equal to the length of the binary encoding of x—note that through this restriction the guessing action is in polynomial time—and define $\omega \in \Omega_{t_{\text{prod}}}$ through $\omega(a_i) = y_i$. Then verify (in polynomial time) that $E^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K},\omega) \leq x$ and therefore $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}) \leq x$. Note that due to Lemma 1 it is sufficient to guess the y_i from the given finite set.

NP-completeness follows from the fact that we can reduce SAT to $\textsc{Upper}_{\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}}$ with $x = 0$. More precisely, let ϕ be an instance of SAT—i. e. the solution to this problem is "yes" iff ϕ is consistent—, then $(\{\phi\},0)$ is an instance of $\textsc{Upper}_{\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}}$ and as $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$ satisfies CO, $(\{\phi\},0)$ is indeed a positive instance iff ϕ is consistent. As SAT is NP-hard, so is $\textsc{Upper}_{\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}}$.

□

Corollary 1. *For* $\mathcal{I} \in \{\mathcal{I}^{\mathit{fuz}}_{t_{\mathit{prod}}}, \mathcal{I}^{\mathit{fuz},\Sigma}_{t_{\mathit{prod}}}, \mathcal{I}^{\mathit{fuz}}_{t_{\min}}, \mathcal{I}^{\mathit{fuz},\Sigma}_{t_{\min}}\}$ *the problem* $\textsc{Lower}_{\mathcal{I}}$ *is* coNP-*complete and* $\textsc{Exact}_{\mathcal{I}}$ *is in* $\mathsf{D}^{\mathsf{P}}_1$.

Proof. Let $n_{\mathcal{K}}$ be the size of $\mathcal{K}$ defined via

$$n_{\mathcal{K}} = \sum_{\alpha\in\mathcal{K}} |\alpha|$$

where $|\alpha|$ is the number of connectives $\neg$, $\wedge$, and $\vee$ appearing in the formula α. As the number of knowledge bases $\mathcal{K}$ of size n or less is finite, so is the number of different inconsistency values on these knowledge bases. Let ϵ_n be the minimal distance of two consecutive inconsistency values, i. e., $|\mathcal{I}(\mathcal{K}_1) - \mathcal{I}(\mathcal{K}_2)| \geq \epsilon_n$ for all $\mathcal{K}_1,\mathcal{K}_2$ of maximal size n. Observe that, $(\mathcal{K},x)$ is a positive instance of $\textsc{Upper}_{\mathcal{I}}$ if and only if $(\mathcal{K},x+\epsilon_{n_{\mathcal{K}}})$ is a negative instance of $\textsc{Lower}_{\mathcal{I}}$, showing that $\textsc{Lower}_{\mathcal{I}}$ is coNP-complete, due to Theorem 11. As $\textsc{Exact}_{\mathcal{I}}$ is the combination of the NP-complete problem $\textsc{Upper}_{\mathcal{I}}$ and the coNP-complete problem $\textsc{Lower}_{\mathcal{I}}$, it is in $\mathsf{D}^{\mathsf{P}}_1$. □

Proposition 4. *Let Ω be a set of S-interpretations and E an S-evaluation function. E is supra-classical wrt. Ω if and only if $\mathcal{I}_{\Omega,E}$ satisfies* **CO**.

Proof. Let E be supra-classical wrt. Ω. If $\mathcal{K}$ is consistent then $\mathcal{I}_{\Omega,E}(\mathcal{K}) = 0$ as there is $\omega \in \Omega$ with $E(\mathcal{K},\omega) = 0$ and $\mathcal{I}_{\Omega,E}(\mathcal{K}) = \min\{E(\mathcal{K},\omega) \mid \omega \in \Omega\}$. If $\mathcal{K}$ is consistent then $\mathcal{I}_{\Omega,E}(\mathcal{K}) > 0$ as there is no $\omega \in \Omega$ with $E(\mathcal{K},\omega) = 0$. $\mathcal{I}_{\Omega,E}$ satisfies CO.

$\mathcal{I}_{\Omega,E}$ satisfy CO. If $\mathcal{K}$ is consistent then from $\mathcal{I}_{\Omega,E}(\mathcal{K}) = 0$ it follows $\min\{E(\mathcal{K},\omega) \mid \omega \in \Omega\} = 0$ so there is $\omega \in \Omega$ with $E(\mathcal{K},\omega) = 0$. If $\mathcal{K}$ is inconsistent then from $\mathcal{I}_{\Omega,E}(\mathcal{K}) > 0$ it follows that there cannot be a $\omega \in \Omega$ with $E(\mathcal{K},\omega) = 0$. So E is supra-classical wrt. Ω. □

Proposition 5. *Let Ω be a set of S-interpretations and E an S-evaluation function.*

1. *If E is normalized then $\mathcal{I}_{\Omega,E}$ satisfies* **NO**.
2. *If E is monotonic then $\mathcal{I}_{\Omega,E}$ satisfies* **MO**.
3. *If E is independent then $\mathcal{I}_{\Omega,E}$ satisfies* **IN**.
4. *If E is dominant then $\mathcal{I}_{\Omega,E}$ satisfies* **DO**.

Proof. Let E be normalized. Then we have $\mathcal{I}_{\Omega,E} = \min\{E(\mathcal{K},\omega) \mid \omega \in \Omega\} \in [0,1]$, so $\mathcal{I}_{\Omega,E}$ satisfies NO. The other proofs are analogous. □

Proposition 6. *Let Ω be a set of S-interpretations and E an S-evaluation function. If Ω is finite and $E(\mathcal{K},\omega)$ can be evaluated in polynomial time for every $\mathcal{K} \in \mathbb{K}$ and $\omega \in \Omega$ then* $\textsc{Upper}_{\mathcal{I}_{\Omega,E}}$ *is in* **NP**. *If additionally E is supra-classical wrt. Ω then* $\textsc{Upper}_{\mathcal{I}_{\Omega,E}}$ *is* **NP***-complete.*

Proof. To show NP membership we sketch a non-deterministic polynomial algorithm. Given an instance $(\mathcal{K}, x)$ we first guess an interpretation $\omega \in \Omega$ (as Ω is finite and its length is a constant wrt. the input, this guess needs only constant time) and then verify in polynomial time $E(\mathcal{K},\omega) \leq x$, proving $\mathcal{I}_{\Omega,E}(\mathcal{K}) \leq x$.

For hardness, observe that $\mathcal{I}_{\Omega,E}(\mathcal{K}) = 0$ iff $\mathcal{K}$ is consistent, so we can directly reduce SAT to $\textsc{Upper}_{\mathcal{I}_{\Omega,E}}$. □

Proposition 7. $\mathcal{I}_{\Omega(\mathsf{At}),E_d^\Sigma} = \mathcal{I}_d^\Sigma$, $\mathcal{I}_{\Omega(\mathsf{At}),E_d^{\max}} = \mathcal{I}_d^{\max}$, $\mathcal{I}_{\Omega(\mathsf{At}),E_d^{hit}} = \mathcal{I}_d^{hit}$.

Proof. We only show $\mathcal{I}_{\Omega(\mathsf{At}),E_d^\Sigma} = \mathcal{I}_d^\Sigma$, the other proofs are analogous.

Let $\mathcal{K} = \{\alpha_1,\ldots,\alpha_n\}$ and $\mathcal{I}_d^\Sigma(\mathcal{K}) = x$. Then there is $(k_1,\ldots,k_n) \in P_d(\mathcal{K})$ with $x = k_1 + \ldots + k_n$. Furthermore, there is $\omega \in M_d^{k_1}(\alpha_1) \cap \ldots \cap M_d^{k_n}(\alpha_n)$ with

$d(\omega, \omega') \leq k_i$ for some $\omega' \in \mathsf{Mod}(\alpha_i)$ and all $i = 1, \ldots, n$. As $k_1 + \ldots + k_n$ is minimal each k_i, $i = 1, \ldots, n$, is minimal as well and we have $d(\omega, \omega') = k_i$ for some $\omega' \in \mathsf{Mod}(\alpha_i)$ and all $i = 1, \ldots, n$. In other words, $d(\mathsf{Mod}(\alpha_i), \omega) = k_i$ and

$$\begin{aligned} x &= k_1 + \ldots + k_n \\ &= d(\mathsf{Mod}(\alpha_1), \omega) + \ldots + d(\mathsf{Mod}(\alpha_n), \omega) \\ &= E^{\sum}(\mathcal{K}, \omega) \end{aligned}$$

showing $\mathcal{I}_d^{\Sigma}(\mathcal{K}) \geq \mathcal{I}_{\Omega(\mathsf{At}), E_d^{\Sigma}}(\mathcal{K})$.

Let now $\mathcal{I}_{\Omega(\mathsf{At}), E_d^{\Sigma}}(\mathcal{K}) = x$. Then there is $\omega \in \Omega(\mathsf{At})$ with $E_d^{\sum}(\mathcal{K}, \omega) = x = d(\mathsf{Mod}(\alpha_1), \omega) + \ldots + d(\mathsf{Mod}(\alpha_n), \omega)$. Therefore, $(d(\mathsf{Mod}(\alpha_1), \omega), \ldots, d(\mathsf{Mod}(\alpha_n), \omega)) \in P_d(\mathcal{K})$ and $\mathcal{I}_d^{\Sigma}(\mathcal{K}) \leq \mathcal{I}_{\Omega(\mathsf{At}), E_d^{\Sigma}}(\mathcal{K})$. $\square$