On Grounding Extensions in Abstract Argumentation

Lars BENGEL [a], Shawn BOWERS [b], Bertram LUDÄSCHER [c], and Matthias THIMM [a,1]

[a] *Artificial Intelligence Group, University of Hagen, Germany*
[b] *Department of Computer Science, Gonzaga University, USA*
[c] *School of Information Sciences, University of Illinois, Urbana-Champaign, USA*

Abstract. The grounded semantics is the baseline semantics for abstract argumentation frameworks as it constitutes the most skeptical outcome of the argumentation. Based on the observation that, for many other semantical notions, any extension can be reduced to the grounded extension of a sub-framework, we investigate conceptual and complexity-theoretic questions on how this *grounding* of extensions behaves. In particular, we identify how minimal sets of attacks (also called *critical attacks*) can be disregarded to obtain a simple representation of possible complex extensions, and provide a full characterisation of computational complexity for six popular semantics. Our approach can particularly be used to explain the inner structure of complex extensions.

1. Introduction

Abstract argumentation frameworks [1] are approaches for modelling argumentative scenarios that represent arguments as nodes in a directed graph, where a directed edge represents an attack from one argument to another. Reasoning in abstract argumentation frameworks consists of identifying sets of arguments (*extensions*) that form a *plausible* outcome of the argumentation. Several different semantical approaches for formalising plausibility in this context have been proposed, see e. g. [1,2]. Central to most of these approaches are the concepts of *conflict-freeness* and *admissibility*, which model the requirements that a plausible outcome should be free of internal conflicts and defend itself against threats from the outside, respectively. Many different semantics have been proposed based on these notions, but also based on different concepts such as rankings [3], weights [4], or probabilities [5].

In this work, the grounded semantics [1] is of special interest. The grounded extension of an abstract argumentation framework is uniquely determined and can be easily defined using a very simple procedure. More precisely, the grounded extension is obtained by first accepting all arguments that are unattacked (i. e., that have no incoming edge), then removing them and all arguments attacked by them, and iterating this procedure until no further arguments can be accepted. Using this procedure, the acceptability of arguments in the grounded extension is also easily explained, as there is no cyclic

[1] Corresponding Author.

dependency between accepted arguments and acceptability of an argument follows naturally from the acceptability of its defenders. The grounded extension can be regarded as the most skeptical (or cautious) outcome of the modelled argumentation. In fact, any extension of most other semantics, in particular complete, preferred, stable, and semi-stable semantics [2], has the grounded extension as a subset. Moreover, it can also easily be seen (see Proposition 2 below), that by removing some attacks of the abstract argumentation framework, any extension of the above-mentioned semantics (and also many more) can be recovered as the grounded extension of the resulting smaller framework. This insight is central for the work presented in this paper. It allows us to reduce a possibly complex (in terms of defense structure) extension to a more simple structure that is easily explainable. Moreover, removed attacks can be regarded as *inactive* attacks, also called *critical attacks* in [6], that do not change the validity of the original extension.

In this paper, we analyse the above setting from a conceptual and complexity-theoretic point of view. In particular, we introduce σ*-critical* sets of attacks that, when removed from the framework, yields that the grounded extension of the resulting framework is a σ-extension of the original framework. We analyse this notion for complete, preferred, stable, and semi-stable semantics and also consider the measure of *grounding distance*, which for a certain σ-extension is the minimal size of a corresponding σ-critical set. As the main contribution, we characterise the computational complexity of certain decision problems in this context. This analysis also shows some interesting differences in the relationships between semantics. While for most settings [7], the complexity of problems for stable and preferred semantics usually differ by one level on the polynomial hierarchy (e. g., for deciding skeptical acceptance), here we observe that deciding on an upper bound for the *grounding distance*, we only differ between the classes NP (for stable semantics) and DP (for preferred semantics).

In summary, the contributions of this paper are as follows.

1. We introduce σ-critical sets of attacks and analyse their properties (Section 3).
2. We introduce *grounding distance* as a quantitative tool and analyse its properties (Section 4).
3. We provide a full complexity characterisation of certain problems in our setting (Section 5).

Moreover, Section 2 recalls necessary preliminaries, Section 6 discusses some related works, and Section 7 concludes.

Omitted proofs can be found in the extended version of this paper[2].

2. Abstract argumentation

An *abstract argumentation framework* F is a tuple $F = (A, R)$ where A is a (finite) set of arguments and R is a relation $R \subseteq A \times A$ [1]. For two arguments $a, b \in A$ the relation aRb means that argument a attacks argument b. For a set $S \subseteq A$ we define

$$S_F^+ = \{a \in A \mid \exists b \in S, bRa\} \qquad S_F^- = \{a \in A \mid \exists b \in S, aRb\}$$

[2] `https://zenodo.org/records/19483871`

If S is a singleton set, i. e., $S = \{a\}$ for some $a \in A$, then we also just write a_F^+ (resp. a_F^-) for $\{a\}_F^+$ (resp. $\{a\}_F^-$).

We say that a set $S \subseteq A$ is *conflict-free* if for all $a, b \in S$ it is not the case that aRb. A set S *defends* an argument $b \in A$ if for all a with aRb there is $c \in S$ with cRa. A conflict-free set S is called *admissible* if S defends all $a \in S$.

Different semantics [2] can be phrased by imposing certain constraints on sets of arguments. In particular, a set S

- is a *complete* (co) extension iff it is admissible and for all $a \in A$, if S defends a then $a \in S$,
- is a *grounded* (gr) extension iff it is complete and minimal,
- is a *stable* (st) extension iff it is conflict-free and $S \cup S_F^+ = A$,
- is a *preferred* (pr) extension iff it is admissible and maximal.
- is a *semi-stable* (sst) extension iff it is complete and $S \cup S_F^+$ is maximal.

All statements on minimality/maximality are meant to be with respect to set inclusion. For a semantics $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{st}, \mathsf{pr}, \mathsf{sst}\}$ let $\sigma(F)$ denote the set of all σ-extensions of $F = (A, R)$. Since the grounded extension is uniquely determined [1], we directly write $\mathsf{gr}(F) = S$ for the grounded extension of F (instead of $\mathsf{gr}(F) = \{S\}$).

For some proofs, it will be convenient to use the following characterisation of the grounded extension, which relies on the notion of the *reduct* [8]. For a framework $F = (A, R)$ and $S \subseteq A$, the *reduct* F^S of F wrt. S is $F^S = (A', R')$ with $A' = A \setminus (S \cup S_F^+)$ and $R' = R \cap (A' \times A')$. Let furthermore $\mathsf{unatt}(F) = \{a \mid a_F^- = \emptyset\}$ be the set of unattacked arguments of F.

Proposition 1. *Let $F = (A, R)$ be an abstract argumentation framework with $n = |A|$. We have*

$$\mathsf{gr}(F) = S_1 \cup \ldots \cup S_n$$

where $S_1 = \mathsf{unatt}(F)$ and $S_i = \mathsf{unatt}(F^{S_1 \cup \ldots \cup S_{i-1}})$ for all $i = 2, \ldots, n$.

In other words, the grounded extension $\mathsf{gr}(F)$ of F can be determined by iteratively adding all unattacked arguments and removing all those arguments and those attacked by them. Note that this characterisation is merely a notational variant of the definition of the grounded extension via the *characteristic function* [1] and is therefore given without proof.

For $A' \subseteq A$ define the *projection* $F|_{A'}$ of F to A' as $F|_{A'} = (A', R \cap (A' \times A'))$. We are also interested in so-called *local deletions* [9], i. e., sub-frameworks of F that are obtained by removing certain attacks. For $R' \subseteq A \times A$ define

$$F - R' = (A, R \setminus R').$$

We also call $F - R'$ the *restriction* of F by R'.

3. Grounding extensions through σ-critical attacks

The central motivation for our work is given by the following observation.

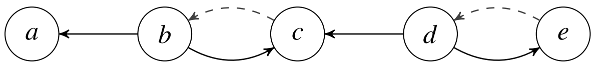

Figure 1. The argumentation framework F_1 from Example 1.

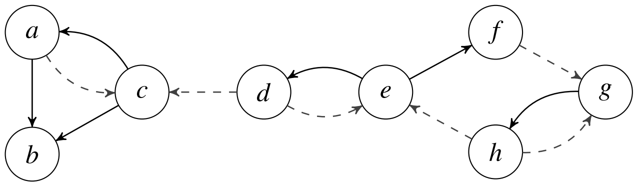

Figure 2. The argumentation framework F_2 from Example 2.

Proposition 2. *Let* $F = (A,R)$ *be an abstract argumentation framework and* $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{st}, \mathsf{pr}, \mathsf{sst}\}$. *If* $S \in \sigma(F)$ *then there is* $R' \subseteq R$ *such that* $S = \mathsf{gr}(F - R')$.

Proof. Define

$$R' = \{(a,b) \in R \mid b \in S\}.$$

Then every argument in S is unattacked in $F - R'$ and therefore $S \subseteq \mathsf{gr}(F - R')$. Assume $S \neq \mathsf{gr}(F - R')$, then there is an argument $c \in \mathsf{gr}(F - R')$ with $c \notin S$, $S \cup \{c\}$ is conflict-free in $F - R'$, and S defends c in $F - R'$. Assume $S \cup \{c\}$ is not conflict-free in F, so there is a $b \in S$ and $(b,c) \in R$ or $(c,b) \in R$:

- Case $(b,c) \in R$: this directly violates the conflict-freeness of $S \cup \{c\}$ in $F - R'$ as $(b,c) \notin R'$.
- Case $(c,b) \in R$: Since $S \in \sigma(F)$, S is admissible in F and must defend b, so there is $b' \in S$ with $(b',c) \in R$. But as $b' \in S$ and the attack (b',c) is also present in $F - R'$, the set $S \cup \{c\}$ would not be conflict-free in $F - R'$.

So $S \cup \{c\}$ is conflict-free in F. As S defends c in $F - R'$, it also defends c in F. So S is not complete in F, contradicting $S \in \sigma(F)$ since $\sigma(F) \subseteq \mathsf{co}(F)$ for all $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{st}, \mathsf{pr}, \mathsf{sst}\}$. It follows $S = \mathsf{gr}(F - R')$. □

In other words, any extension $S \in \sigma(F)$ can be defined as the grounded extension of a certain restriction $F - R'$ of F. However, note that there may be multiple R' for which this statement may be true (see also below).

We illustrate the observation from Proposition 2 with some examples.

Example 1. We consider the abstract argumentation framework F_1 in Figure 1, with $\sigma(F_1) = \{\{b,d\}, \{b,e\}, \{a,c,e\}\}$ for $\sigma \in \{\mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$. We take $S = \{b,d\}$ and let $R' = \{(c,b), (e,d)\}$. The restriction $F_1 - R'$ is depicted in Figure 1, if attacks indicated in gray and a dashed line are disregarded. It is then easy to see that $\mathsf{gr}(F_1 - R') = S$.

Example 2. Examine the abstract argumentation framework F_2 in Figure 2. We consider $S = \{c,e,g\}$ with $S \in \sigma(F_2)$ and $\sigma \in \{\mathsf{co}, \mathsf{st}, \mathsf{pr}, \mathsf{sst}\}$. Let $R' = \{(a,c), (d,c), (d,e), (f,g), (h,g), (h,e)\}$. The restriction $F_2 - R'$ is also shown in Figure 2, with the attacks in R' indicated in dark gray and a dashed line. We then have that $\mathsf{gr}(F_2 - R') = S$.

Observe that the other direction of the statement in Proposition 2 (i. e., from $S = \mathsf{gr}(F - R')$ for some R' it follows that $S \in \sigma(F)$) is not true in general (consider removing all attacks). Following [6], we call sets R' for which the statement holds *σ-critical*.

Definition 1. Let $F = (A,R)$ be an abstract argumentation framework and $\sigma \in \{\mathsf{co},\mathsf{gr},\mathsf{st},\mathsf{pr},\mathsf{sst}\}$. A set $R' \subseteq R$ is *σ-critical* for F if $\mathsf{gr}(F - R') \in \sigma(F)$.

In other words, a set of attacks R' is *σ-critical* if the grounded extension of $F - R'$ is a σ-extension of F. If $S = \mathsf{gr}(F - R')$ for $S \in \sigma(F)$ we also say that S is *grounded* by R'.

Example 3. We consider again the abstract argumentation framework F_1 in Figure 1. We have that $R' = \{(c,b),(e,d)\}$ is σ-critical for F_1 and $\sigma \in \{\mathsf{co},\mathsf{st},\mathsf{pr},\mathsf{sst}\}$. Note that $R'' = \{(c,b),(e,d),(b,c)\}$ is also σ-critical with $\mathsf{gr}(F_1 - R'') = \{b,d\}$. Similarly, even $R''' = \{(e,d)\}$ is σ-critical for F_1 and the same semantics.

We identify some simple properties of critical attacks that are used later.

Proposition 3. *Let $F = (A,R)$ be an abstract argumentation framework, $R' \subseteq R$, and $\sigma \in \{\mathsf{co},\mathsf{gr},\mathsf{st},\mathsf{pr},\mathsf{sst}\}$.*

1. *$\emptyset$ is gr-critical and co-critical for F.*
2. *If $\sigma(F) \subseteq \sigma'(F)$ and R' is σ-critical then R' is σ'-critical.*
3. *for $\sigma \neq \mathsf{co}$, $\emptyset$ is σ-critical in F iff $\sigma(F) = \mathsf{gr}(F)$.*

As illustrated in Example 3 there may be multiple sets R' and R'' such that R' and R'' are both σ-critical and we also have $\mathsf{gr}(F - R') = \mathsf{gr}(F - R'')$. Naturally, we are interested in minimal modifications of F in order to obtain a grounding of a certain extension $S \in \sigma(F)$.

Definition 2. Let $F = (A,R)$ be an abstract argumentation framework and $\sigma \in \{\mathsf{co},\mathsf{gr},\mathsf{st},\mathsf{pr},\mathsf{sst}\}$. A set $R' \subseteq R$ is *minimally σ-critical* for F, if R' is σ-critical for F and for all R'' with $R'' \subsetneq R' \subseteq R$, R'' is not *σ-critical* for F.

In the above definition, we consider minimality in terms of subset minimality, which appears to be the most natural choice to us. However, in the next section we will also consider the case of minimality wrt. cardinality.

Example 4. We continue Example 3 with the abstract argumentation framework F_1 in Figure 1. We have that $R''' = \{(e,d)\}$ is minimally σ-critical for F_1 and $\sigma \in \{\mathsf{st},\mathsf{pr},\mathsf{sst}\}$. The other minimally σ-critical sets for F_1 are $R' = \{(b,c),(d,e)\}$ and $R'' = \{(c,b),(d,e)\}$, where we obtain $\mathsf{gr}(F_1 - R') = \{a,c,e\}$ and $\mathsf{gr}(F_1 - R'') = \{b,e\}$, respectively. On the other hand, $\emptyset$ is the only minimally σ'-critical set for $\sigma' \in \{\mathsf{co},\mathsf{gr}\}$.

We can observe that attacks in a minimal σ-critical set are necessarily attacks onto the extension under consideration.

Proposition 4. *Let $F = (A,R)$ be an abstract argumentation framework and $\sigma \in \{\mathsf{co},\mathsf{gr},\mathsf{pr},\mathsf{st},\mathsf{sst}\}$. If R' is minimal σ-critical and $S = \mathsf{gr}(F - R')$, then for all $(a,b) \in R'$ we have $a \in S^-_F \cap S^+_F$ and $b \in S$.*

In other words, every attack in a σ-critical set is an attack on the extension that is also defended against by the extension.

Example 5. We continue Example 4 with the abstract argumentation framework F_1 in Figure 1. If we consider any of the minimally pr-critical sets for F_1, we notice that every attack is either between b and c or d and e. More specifically, if, for instance, the attack (d,e) is included in the minimally pr-critical set, then the argument e is contained in the resulting grounded extension. The same holds for all other attacks in minimally pr-critical sets.

The following results dive a bit deeper into the inner structure of σ-critical sets. In particular, we show that (except for complete semantics) subsets of σ-critical sets that are also σ-critical always ground the same extension (Lemma 1). Moreover, the lattice over the subset relation of σ-critical sets is *smooth*, i. e., any set 'between' two σ-critical sets is also σ-critical (see Lemma 2). This allows us to obtain a simple characterisation (Proposition 5) of σ-critical sets that will be useful later for our complexity analysis.

Lemma 1. *Let $F=(A,R)$ be an abstract argumentation framework, $R' \subseteq R'' \subseteq R$, and $\sigma \in \{\mathsf{pr},\mathsf{st},\mathsf{sst},\mathsf{gr}\}$. If R' and R'' are σ-critical then* $\mathsf{gr}(F-R')=\mathsf{gr}(F-R'')$.

Proof. First observe that the statement is trivially satisfied for $\sigma=\mathsf{gr}$ since the grounded extension is uniquely determined and we have $\mathsf{gr}(F-R')=\mathsf{gr}(F)=\mathsf{gr}(F-R'')$.

Let $n=|A|$ and

$$\mathsf{gr}(F-R') = S'_1 \cup \ldots \cup S'_n$$
$$\mathsf{gr}(F-R'') = S''_1 \cup \ldots \cup S''_n$$

as in Proposition 1. Assume $\mathsf{gr}(F-R') \neq \mathsf{gr}(F-R'')$. Since σ satisfies I-Maximality [10], we have neither $\mathsf{gr}(F-R') \subsetneq \mathsf{gr}(F-R'')$ nor $\mathsf{gr}(F-R') \supsetneq \mathsf{gr}(F-R'')$, so we have in particular $\mathsf{gr}(F-R') \setminus \mathsf{gr}(F-R'') \neq \emptyset$. Let $i \in \{1,\ldots,n\}$ be the smallest index such that there is $a \in S'_i$ but $a \notin \mathsf{gr}(F-R'')$. Consider the following cases:

1. Assume there is $b \in \mathsf{gr}(F-R'')$ with $(b,a) \in R \setminus R''$. Since $(b,a) \in R \setminus R''$ we also have $(b,a) \in R \setminus R'$ (due to $R' \subseteq R''$). As $a \in S'_i$, a is unattacked in $(F-R')^{S'_1 \cup \ldots \cup S'_{i-1}}$. If $i=1$ then necessarily $a \in S''_1$ as well (a is also unattacked in $F-R''$ and therefore $a \in \mathsf{gr}(F-R'')$), so we have $i>1$. Then there must be a $c \in S'_{i-1}$ with $(c,b) \in R \setminus R'$ and therefore $(c,b) \in R$. Since $\mathsf{gr}(F-R'')$ is conflict-free, it follows $c \notin \mathsf{gr}(F-R'')$. This is a contradiction to the assumption that i is the smallest index such that there is $a \in S'_i$ but $a \notin \mathsf{gr}(F-R'')$ (since $c \in S'_{i-1}$ and $c \notin \mathsf{gr}(F-R'')$).
2. Assume there is no $b \in \mathsf{gr}(F-R'')$ with $(b,a) \in R \setminus R''$. Since i is the smallest index such that there is $a \in S'_i$ but $a \notin \mathsf{gr}(F-R'')$ we have

$$S'_1 \cup \ldots \cup S'_{i-1} \subseteq \mathsf{gr}(F-R'')$$

Since $a \in S'_i$, a is unattacked in $(F-R')^{S'_1 \cup \ldots S'_{i-1}}$. As $(F-R'')$ has even less attacks than $(F-R')$, a is also unattacked in $(F-R')^{\mathsf{gr}(F-R'')}$ and $\mathsf{gr}(F-R'') \cup \{a\}$ is conflict-free. So $\mathsf{gr}(F-R'')$ cannot be grounded and we arrived at a contradiction.

It follows $\mathsf{gr}(F-R')=\mathsf{gr}(F-R'')$. □

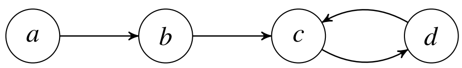

Figure 3. The argumentation framework F_3 from Example 6.

Note that the above statement can be generalised to any semantics that satisfies I-Maximality [10]. However, observe that the above statement is not satisfied for the complete semantics (which also does not satisfy I-Maximality).

Example 6. We consider the abstract argumentation framework F_3 in Figure 3. We have that $\mathsf{co}(F_3) = \{\{a\},\{a,c\},\{a,d\}\}$. Consider, for instance, $R' = \emptyset$ and $R'' = \{(b,c),(d,c)\}$, which are both co-critical for F_3 and we have $R' \subseteq R''$. However, it is easy to see that $\mathsf{gr}(F_3 - R') = \{a\} \neq \{a,c\} = \mathsf{gr}(F_3 - R'')$. In contrast to that, R' is not σ-critical for F_3 with $\sigma \in \{\mathsf{pr},\mathsf{st},\mathsf{sst}\}$ while R'' is not gr-critical for F_3.

Lemma 2. *Let $F = (A,R)$ be an abstract argumentation framework, $R' \subseteq R''' \subseteq R'' \subseteq R$, and $\sigma \in \{\mathsf{pr},\mathsf{st},\mathsf{sst},\mathsf{gr}\}$. If R' and R'' are σ-critical then R''' is σ-critical.*

For the proof of the above lemma, see the extended version of this paper[3].

From Lemma 1 and Lemma 2 we can derive the following characterisation for minimial σ-critical sets, which basically says that imposing minimality computationally only adds polynomial overhead.

Proposition 5. *Let $F = (A,R)$ be an abstract argumentation framework, $R' \subseteq R$, and $\sigma \in \{\mathsf{pr},\mathsf{st},\mathsf{sst},\mathsf{gr}\}$, The following two statements are equivalent:*

1. *R' is minimally σ-critical.*
2. *R' is σ-critical and for all $(a,b) \in R'$, $R' \setminus \{(a,b)\}$ is not σ-critical.*

The notion of (minimal) σ-critical sets allows us to explain the acceptability of arguments in an arbitrary extension in a procedural manner. More precisely, if R' is minimally σ-critical and grounds $S \in \sigma(F)$, then we can first set aside the attacks R' and explain why $S = \mathsf{gr}(F - R')$ by following the characterisation of Proposition 1: some arguments of S are unattacked, they are accepted right away. Then arguments attacked by already accepted arguments can be safely removed since they are defeated. We iterate this procedure until no more arguments can be accepted. Then we attend again the attacks in R' and see (thanks to Proposition 4) that these are attacks from arguments defeated through the previous process and thus of no consequence to the acceptability of the arguments in S.

Example 7. We consider again the abstract argumentation framework F_2 from before, reproduced in Figure 4, and the preferred extension $S = \{c,e,g\}$. We examine the minimally pr-critical set $R' = \{(a,c),(d,e),(h,g),(f,g)\}$ (again indicated in gray), which allows us to explain the acceptability of S as follows. Initially, only the argument g is unattacked. After removing the defeated argument h, the argument e is now unattacked and can be accepted. Finally, due to the removal of the defeated argument d, the argument c can be accepted, and no further arguments may be accepted.

[3] `https://zenodo.org/records/19483871`

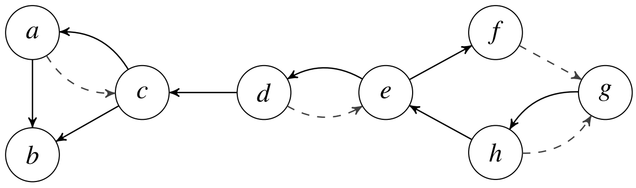

Figure 4. The argumentation framework F_2 and the restriction $F_2 - R'$ from Example 7.

Alternatively, the set $R'' = \{(a,c),(d,e),(h,g),(h,e)\}$ is also minimally pr-critical for S in F_2. Under this critical set, we first accept e and, after the removal of d and f, can accept both c and g in the second step.

4. Grounding distance

The notion of σ-critical sets allows us also to *measure* the effort required for grounding extensions quantitatively.

Definition 3. Let $F = (A,R)$ be an abstract argumentation framework, $\sigma \in \{\mathsf{co},\mathsf{gr},\mathsf{st},\mathsf{pr},\mathsf{sst}\}$, and $S \in \sigma(F)$. The *grounding distance* $\mathsf{gdist}_F^\sigma(S)$ of S in F wrt. σ is defined as

$$\mathsf{gdist}_F^\sigma(S) = \min\{|R'| \mid R' \text{ is } \sigma\text{-critical for } F \text{ and } S = \mathsf{gr}(F - R')\}$$

with $\min\emptyset = \infty$.

In other words, $\mathsf{gdist}_F^\sigma(S)$ is the minimal size of σ-critical set that grounds S (here now in terms of set cardinality). It therefore captures how close S is to being grounded and can therefore be used as a measure of the inner complexity (in terms of defense structure) of S.

Example 8. We continue Example 4 with the abstract argumentation framework F_1 in Figure 1 and consider $\sigma \in \{\mathsf{st},\mathsf{pr},\mathsf{sst}\}$. For the σ-extension $S = \{b,d\}$, we have $\mathsf{gdist}_F^{\mathsf{pr}}(S) = 1$. On the other hand, we also have $\mathsf{gdist}_F^\sigma(\{b,e\}) = \mathsf{gdist}_F^\sigma(\{a,c,e\}) = 2$. Finally, for the grounded extension $\mathsf{gr}(F_1) = \emptyset$, we obtain $\mathsf{gdist}_F^{\mathsf{gr}}(\emptyset) = 0$. Note that for any other set $S' \subseteq A$, we get a grounding distance of ∞.

Some simple observations on the grounding distance are as follows.

Proposition 6. *Let $F = (A,R)$ be an abstract argumentation framework, $\sigma \in \{\mathsf{co},\mathsf{gr},\mathsf{st},\mathsf{pr},\mathsf{sst}\}$, and $S \subseteq A$.*

1. *$\mathsf{gdist}_F^\sigma(S) \neq \infty$ iff $S \in \sigma(F)$.*
2. *$\mathsf{gdist}_F^{\mathsf{gr}}(\mathsf{gr}(F)) = 0$.*

The grounding distance can be used to compare different σ-extensions and it allows finer-grained reasoning by considering only those σ-extensions with 'small' grounding distance. In particular, this measure can be used as a selection criterion for the approach of [11], which uses such selection criterions to filter out some σ-extensions and defines a variant of skeptical reasoning taking only the remaining σ-extensions into account, see also [12] for a similar work in this direction. We leave a deeper investigation of this aspect for future work.

5. Computational complexity and algorithms

We assume familiarity with basic concepts of computational complexity and basic complexity classes such as P, NP, coNP, see [13] for an introduction. For two complexity classes $\mathscr{C}$ and $\mathscr{D}$, the class $\mathscr{C}^{\mathscr{D}}$ is the class of decision problems solvable by an algorithm in class $\mathscr{C}$ that has access to an *oracle* of class $\mathscr{D}$ (which means that the algorithm can obtain the answer of any problem in class $\mathscr{D}$ within one step). The class $\mathsf{P}_{\|}^{\mathsf{NP}}$ is the class of decision problems solvable by a deterministic polynomial-time algorithm that can make polynomially many *parallel* (or *non-adaptive*) calls to an NP-oracle (so results of the calls cannot influence the problems solved by other calls). Note that $\mathsf{P}_{\|}^{\mathsf{NP}} = \mathsf{P}^{\mathsf{NP}[\log]}$, where $\mathsf{P}^{\mathsf{NP}[\log]}$ is the class of decision problems solvable by a deterministic polynomial-time algorithm that can make only logarithmically many calls to an NP-oracle, but which can be *adaptive* (i. e., the result of one oracle call can influence the problem definition for the next oracle call). Finally, we require knowledge of the class DP, which is defined via $\mathsf{DP} = \{L_1 \cap L_2 \mid L_1 \in \mathsf{NP}, L_2 \in \mathsf{coNP}\}$. In other words, DP is the class of problems that are intersections of NP and coNP problems (but note that, under standard complexity-theoretic assumptions, $\mathsf{DP} \neq \mathsf{NP} \cap \mathsf{coNP}$).

We consider the following computational tasks:

VERCRITICAL$_\sigma$	**Input**:	$F = (A,R)$, $R' \subseteq R$
	Output:	YES iff R' is σ-critical
VERMINCRITICAL$_\sigma$	**Input**:	$F = (A,R)$, $R' \subseteq R$
	Output:	YES iff R' is minimally σ-critical
DISTUPPER$_\sigma$	**Input**:	$F = (A,R)$, $S \subseteq A$, $k \in \mathbb{N}$
	Output:	YES iff $\mathsf{gdist}_F^\sigma(S) \leq k$

Informally, the problem VERCRITICAL is about verifying whether a set of attacks is σ-critical, wrt. some given semantics σ. Similarly, the problem VERMINCRITICAL is about verifying whether a set of attacks is minimally σ-critical. The problem DISTUPPER is about checking whether some integer k is an upper bound for the grounding distance of a set of arguments S.

In order to characterise the complexity of some of the above problems, the following (quite straightforward) observation will be helpful.

Lemma 3. *Let F be an abstract argumentation framework. Deciding whether $\sigma(F) = \mathsf{gr}(F)$ is coNP-complete for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$.*

We can now provide a complete complexity characterisation of the above problems.

Theorem 1. *Let $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{st}, \mathsf{pr}, \mathsf{sst}\}$. The computational complexities for the problems* VERCRITICAL$_\sigma$*,* VERMINCRITICAL$_\sigma$*, and* DISTUPPER$_\sigma$ *are as shown in Table 1.*

6. Related works

The notion of *critical attacks* has been introduced in [6], which, first and foremost, analyses this notion from a conceptual point of view and its relationships to concepts from

σ	VERCRITICAL$_\sigma$	VERMINCRITICAL$_\sigma$	DISTUPPER$_\sigma$
co	in P	trivial	NP-c
gr	in P	trivial	in P
st	in P	in P	NP-c
pr	coNP-c	coNP-c	DP-c
sst	coNP-c	coNP-c	DP-c

Table 1. Computational complexities of the problems VERCRITICAL$_\sigma$, VERMINCRITICAL$_\sigma$, and DISTUPPER$_\sigma$ for $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{st}, \mathsf{pr}, \mathsf{sst}\}$.

game theory. In particular, [6] considers the *Skeptic's argumentation game*, which is an application of *win-move* programs for solving two-player games to abstract argumentation frameworks. The solution of these programs is equivalent to the grounded extension and [6] identifies (among other things) critical attacks as relevant for finding the *provenance* of certain choices made during the game. However, their investigation does not touch those aspects we considered in this paper, in particular, they did not conduct an analysis of computational complexity.

Also related to our work is the area of *extension enforcement* [9], in particular for grounded semantics [14,15]. The computational problem addressed here is given an argumentation framework $F = (A, R)$ and a set $S \subseteq A$, how to minimally 'change' F to F' such that S is an extension of F'. While the literature on enforcement considers a wide variety on different change operations, our setting fits the scenario of grounded extension enforcement under local deletions [9]. However, there are certain noteworthy differences to our setting. First, we only consider enforcement of sets S that are already σ-extensions (so our scenario always considers two semantics: σ and the grounded semantics). Moreover, we are more interested in the critical sets and their role in 'grounding extensions'. This has direct influence on our computational complexity analysis, which is completely orthogonal to the computational complexity of grounded enforcement [14].

Another related field are explanations in formal argumentation. Various works are concerned with the problem of explaining the acceptance [16,17,18] or rejection of individual arguments [19,20,21]. For instance, in [22] sufficient and necessary reasons for the acceptance or rejection of arguments are considered. Complementary to that, there are also approaches that consider explaining sets of arguments [23,24,25]. Related to that, in [26] a scheme for recursively constructing extensions along the strongly connected components (SCCs) of an argumentation framework is introduced. In [27], this idea is then used to construct structural explanations for acceptance in probabilistic argumentation frameworks by constructing σ-extensions according to the SCC-structure. Notably, the above mentioned works are usually focused on the arguments in the argumentation framework, while we are more interested in the attacks that are critical in 'grounding' an extension.

7. Summary and conclusion

We considered the problem of explaining the inner structure of complex extensions by reducing them to the grounded extension of an appropriate sub-framework. For that we analysed (minimal) *critical sets* of attacks, i. e., attacks that can be removed from a framework in order to obtain equivalence of the given extension with the grounded extension.

We also introduced the notion of *grounding distance* to quantitively measure the effort required to ground extensions. Our main contribution is a comprehensive complexity analysis of certain decision problems wrt. six argumentation semantics.

For future work, we intend to investigate the potential of using the grounding distance in the context of *gradual evaluation* of argument strength [28,29,30], and in particular for relating sets of arguments wrt. their strength [11,12] as discussed at the end of Section 4. Moreover, following up on the work [6], another avenue for future work is to develop graphical tools for explaining argument acceptance through visualising the role of critical attacks and the grounding of extensions.

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