Optimisation and Approximation in Abstract Argumentation: The Case of Stable Semantics

Proofs of technical results

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Proposition 1. *Let $F = (A, R)$ be an AF, $k \in \mathbb{N}$, $S \subseteq A$, and $a \in A$.*

1. *S is a $|A|$-stable extension iff S is a stable extension.*
2. *If S is a k-stable extension then S is a k'-stable extension for all $k' < k$.*
3. *$S \in \mathsf{cf}(F)$ iff S is a 0-stable extension.*
4. *$(a, a) \notin R$ iff $\{a\}$ is a 1-stable extension.*
5. *$\emptyset$ is a 0-stable extension.*

Proof.

1. If S is a $|A|$-stable extension, it follows $|S^{\oplus}_F| \geq |A|$ and since $S \cap S^{+}_F = \emptyset$ we have $S^{\oplus}_F = A$ and S is a stable extension. The other direction is immediate as well.
2. This follows directly via $|S^{\oplus}_F| \geq k > k'$.
3. If S is conflict-free then $|S^{\oplus}_F| \geq 0$ naturally. Since every k-stable extension is conflict-free, the other direction follows as well.
4. If $(a, a) \notin R$ then $\{a\}$ is conflict-free and $|\{a\}^{\oplus}_F| \geq 1$ due to $a \in \{a\}$. For the other direction, since $\{a\}$ must be conflict-free, $(a, a) \notin R$ follows directly.
5. $\emptyset$ is conflict-free and satisfies $|\emptyset^{\oplus}_F| \geq 0$. □

Theorem 2. *Let $k \in \mathbb{N}$ be an additional input parameter to the following problems.* $\textsc{Ver}_{\mathsf{st}_k}$ *is in P,* $\textsc{Cred}_{\mathsf{st}_k}$ *is NP-complete,* $\textsc{Skep}_{\mathsf{st}_k}$ *is coNP-complete,* $\textsc{Exists}_{\mathsf{st}_k}$ *is NP-complete,* $\textsc{Exists}^{\neg\emptyset}_{\mathsf{st}_k}$ *is NP-complete.*

Proof. Note that all hardness results following directly from the fact that the corresponding problems for stable semantics are hard wrt. these classes and the problems for k-stable semantics are generalisations of the corresponding problems for stable semantics (e. g., checking whether there exists a stable extension in $F = (A, R)$ is equivalent to checking whether there exists a $|A|$-stable extension in F). So only the membership parts remain to be shown. Let $F = (A, R)$ be an AAF.

- $\textsc{Ver}_{\mathsf{st}_k} \in \mathsf{P}$:
  Given $S \subseteq A$, we can verify both conflict-freeness of S and the fact that $|S^{\oplus}_F| \geq k$ in polynomial time.
- $\textsc{Cred}_{\mathsf{st}_k} \in \mathsf{NP}$:
  On input $a \in A$, we non-deterministically guess a set $S \subseteq A$ with $a \in S$ and verify in polynomial time that S is a k-stable extension. This shows in NP that a is credulously accepted wrt. k-stable semantics.
- $\textsc{Skep}_{\mathsf{st}_k} \in \mathsf{coNP}$:
  On input $a \in A$, we non-deterministically guess a set $S \subseteq A$ with $a \notin S$ and verify in polynomial time that S is a k-stable extension. This shows in NP that a is *not* skeptically accepted wrt. k-stable semantics. It follows $\textsc{Skep}_{\mathsf{st}_k} \in \mathsf{coNP}$.
- $\textsc{Exists}_{\mathsf{st}_k} \in \mathsf{NP}$:
  We non-deterministically guess a set $S \subseteq A$ and verify in polynomial time that S is a k-stable extension. This shows in NP that there is a k-stable extension.
- $\textsc{Exists}^{\neg\emptyset}_{\mathsf{st}_k} \in \mathsf{NP}$:
  We non-deterministically guess a set $S \subseteq A$ with $S \neq \emptyset$ and verify in polynomial time that S is a k-stable extension. This shows in NP that there is a k-stable extension. □

Proposition 2. *Let $F = (A, R)$ be an AF, $k \in \mathbb{N}$, $S \subseteq A$, and $a \in A$.*

1. *S is a $|R|$-stable* extension iff S is a stable extension.*
2. *If S is a k-stable* extension then S is a k'-stable* extension for all $k' < k$.*
3. *A is a 0-stable* extension.*
4. *If $a \in A$ is not attacked in F, then $a \in S$ for every k-stable* extension S.*

Proof.

1. If S is a $|R|$-stable* extension, it follows that S is conflict-free and since $S \cup S^{+}_F = A$ we have that S is a stable extension. The other direction is immediate as well.
2. This follows directly via $|S^{\circledast}_F| \geq k > k'$.
3. A has clearly full range and due to $|S^{\circledast}_F| \geq 0$ we have that A is a 0-stable* extension.
4. If S is a k-stable extension then either $a \in S$ or $a \in S^{+}_F$ for every $a \in A$. If a is not attacked by any argument, it follows $a \in S$. □

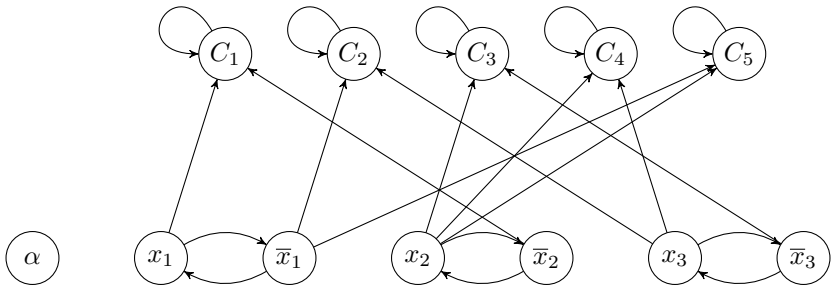


Figure 1: The AAF $F_{\mathcal{C}}$ for $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5\}$ with $C_1 = \{x_1, \overline{x}_2\}$, $C_2 = \{\overline{x}_1, x_3\}$, $C_3 = \{x_2, \overline{x}_3\}$, $C_4 = \{x_2, x_3\}$, $C_5 = \{\overline{x}_1, x_2\}$.

Theorem 3. *Let $k \in \mathbb{N}$ be an additional input parameter to the following problems.* $\text{VER}_{st^*_k}$ *is in P,* $\text{CRED}_{st^*_k}$ *is NP-complete,* $\text{SKEP}_{st^*_k}$ *is coNP-complete,* $\text{EXISTS}_{st^*_k}$ *is NP-complete,* $\text{EXISTS}^{\neg\emptyset}_{st^*_k}$ *is NP-complete.*

Proof. The proof is completely analogous to the proof of Theorem 2. □

Proposition 3. *Let $F = (A, R)$ be an AAF and $a \in A$. If there is a stage extension S with $a \in S$ then $\mathsf{mst}(F, a) \geq |S^{\oplus}_F|$.*

Proof. Let S be a stage extension with $a \in S$. Then S is also a $|S^{\oplus}_F|$-stable extension and therefore $\mathsf{mst}(F, a) \geq |S^{\oplus}_F|$. □

Theorem 4. MAXSTABLE *is $\mathsf{FP}^{\mathsf{NP}[\log]}$-complete.*

Proof. We show $\mathsf{FP}^{\mathsf{NP}[\log]}$-membership by sketching an algorithm that solves MAXSTABLE and runs in (deterministic) polynomial time with at most a logarithmic number of calls to an NP-oracle. Let $F = (A, R)$ and $a \in A$ be the input. We first check whether a attacks itself and return $-\infty$ in that case. Otherwise we perform a binary search for the maximal k with $a \in \bigcup \mathsf{st}_k(F)$ within the range $k \in \{0, \ldots, |A|\}$. More precisely, we start with $k_0 = \lceil |A|/2 \rceil$ and use an NP-oracle (see Theorem 2) to check whether a is credulously inferred with st_{k_0}. If this is false then we continue the search in the range $\{0, \ldots, \lceil |A|/2 \rceil - 1\}$. Otherwise we continue the search in the range $\{\lceil |A|/2 \rceil + 1, \ldots, |A|\}$. After at most logarithmically many search steps, we have identified the maximal k with $a \in \bigcup \mathsf{st}_k(F)$.

In order to show $\mathsf{FP}^{\mathsf{NP}[\log]}$-hardness, we give a reduction from the $\mathsf{FP}^{\mathsf{NP}[\log]}$-complete problem MAXSATSIZE [Papadimitriou, 1994] defined as

MAXSATSIZE
Input a set $\mathcal{C} = \{C_1, \ldots, C_n\}$ of clauses
Output $\max |\{\mathcal{C}' \subseteq \mathcal{C} \mid \mathcal{C}' \not\models \bot\}|$

In other words, on input $\mathcal{C} = \{C_1, \ldots, C_n\}$, the solution to MAXSATSIZE is the maximal number of clauses of $\mathcal{C}$ that can satisfied at once. Let $x_1, \ldots, x_m$ be the atoms appearing in $\mathcal{C}$. We represent $\mathcal{C}$ as an AAF $F_{\mathcal{C}} = (A_{\mathcal{C}}, R_{\mathcal{C}})$—derived from similar translations for other abstract argumentation problems [Dvořák and Dunne, 2018]—with (let α be a new symbol)

$$A_{\mathcal{C}} = \{\alpha, x_1, \ldots, x_m, \overline{x}_1, \ldots, \overline{x}_m\} \cup \{C_1, \ldots, C_n\}$$
$$R_{\mathcal{C}} = \{(x_i, \overline{x}_i), (\overline{x}_i, x_i) \mid i = 1, \ldots, m\} \cup$$
$$\{(\dot{x}, C_j) \mid \dot{x} \in \{x, \overline{x}\}, \dot{x} \in C_j\} \cup$$
$$\{(C_1, C_1), \ldots, (C_n, C_n)\}$$

An example of the above construction for $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5\}$ with $C_1 = \{x_1, \overline{x}_2\}$, $C_2 = \{\overline{x}_1, x_3\}$, $C_3 = \{x_2, \overline{x}_3\}$, $C_4 = \{x_2, x_3\}$, $C_5 = \{\overline{x}_1, x_2\}$ can be seen in Figure 1. We show that $l^* \in \mathbb{N}$ is the maximal number of jointly satisfiable clauses of $\mathcal{C}$ iff l^* is the maximal number with $\alpha \in \bigcup \mathsf{st}_{l^*+2m+1}(F_{\mathcal{C}})$. For that it suffices to show that for any number l, if there are l clauses that can be jointly satisfied then $\alpha \in \bigcup \mathsf{st}_{l+2m+1}(F_{\mathcal{C}})$, and vice versa. We show both directions:

- "⇒": Assume that l clauses can be jointly satisfied and, w.l.o.g., let these be $C_1, \ldots, C_l$. Let ω be a satisfying assignment of $C_1, \ldots, C_l$. Then define the set of arguments S via
$$S = \{\alpha\} \cup \{x_i \mid \omega(x_i) = 1\} \cup \{\overline{x}_i \mid \omega(x_i) = 0\}$$
We show that S is an $(l+2m+1)$-stable extension. First, note that S is conflict-free: α does not attack and is not attacked by any argument, and for each $i = 1, \ldots, m$, we have either $x_i \in S$ or $\overline{x}_i \in S$. Also due to the previous observation, we have that $\alpha, x_1, \overline{x}_1, \ldots x_m, \overline{x}_m \in S \oplus +_{F_{\mathcal{C}}}$. Moreover, since clauses $C_1, \ldots, C_l$ are satisfied by ω, we have that at least one argument of S attacks one of those, so $C_1, \ldots, C_l \in S^{+}_{F_{\mathcal{C}}}$. So we have that all but (potentially) $C_{l+1}, \ldots, C_n$ are in $S^{\oplus}_{F_{\mathcal{C}}}$, so S is a $(l + 2m + 1)$-stable extension.
- "⇐": Assume $\alpha \in \bigcup \mathsf{st}_{l+2m+1}(F_{\mathcal{C}})$, so let S be a $(l + 2m + 1)$-stable extension with $\alpha \in S$. Since S is conflict-free, we have $C_1, \ldots, C_n \notin S$. Moreover, we can assume that for each $i = 1, \ldots, m$ either $x_i \in S$ or $\overline{x}_i \in S$: both cannot be in S without violating conflict-freeness and if neither is in S, we can add x_i without violating the fact that S is a $(l+2m+1)$-stable extension. Since $\alpha \in S$ as well, this accounts for $2m+1$ arguments in $S \oplus +_{F_{\mathcal{C}}}$. Therefore, since S is a $(l + 2m + 1)$-stable extension, at least l of $C_1, \ldots, C_n$ must be attacked by S. Without loss of generality, assume $C_1, \ldots, C_l$ are attacked by S. Define an assignment ω via $\omega(x_i) = 1$ if

$x_i \in S$ and $\omega(x_i) = 0$ if $\overline{x}_i \in S$. It can be seen that ω is a satisfying assignment for $C_1, \ldots, C_l$. □

Theorem 5. MAXSTABLE* *is* $\mathsf{FP}^{\mathsf{NP}[\log]}$*-complete.*

Proof. We show $\mathsf{FP}^{\mathsf{NP}[\log]}$-membership by sketching an algorithm that solves MAXSTABLE* and runs in (deterministic) polynomial time with at most a logarithmic number of calls to an NP-oracle (similar as we did in the proof of Theorem 4). Let $F = (A, R)$ and $a \in A$ be the input. We perform a binary search for the maximal k with $a \in \bigcup \mathsf{st}^*_k(F)$ within the range $k \in \{0, \ldots, |R|\}$. More precisely, we start with $k_0 = \lceil |R|/2 \rceil$ and use an NP-oracle (see Theorem 3) so check whether a is credulously inferred with $\mathsf{st}^*_{k_0}$. If this is false then we continue the search in the range $\{0, \ldots, \lceil |R|/2 \rceil - 1\}$. Otherwise we continue the search in the range $\{\lceil |R|/2 \rceil + 1, \ldots, |R|\}$. After at most logarithmically many search steps, we have identified the maximal k with $a \in \bigcup \mathsf{st}^*_k(F)$.

In order to show $\mathsf{FP}^{\mathsf{NP}[\log]}$-hardness, we give a reduction from the $\mathsf{FP}^{\mathsf{NP}[\log]}$-complete problem MAXSATSIZE [Papadimitriou, 1994], see the proof of Theorem 4 for the definition. The construction is a bit more involved than the construction from the proof of Theorem 4 though. Let $x_1, \ldots, x_m$ be the atoms appearing in the input $\mathcal{C} = \{C_1, \ldots, C_n\}$. We represent $\mathcal{C}$ as an AAF $F^*_\mathcal{C} = (A^*_\mathcal{C}, R^*_\mathcal{C})$ (let all arguments of the type θ and the argument α be fresh):

$$\begin{aligned}
A^*_\mathcal{C} = {} & \{\alpha, x_1, \ldots, x_m, \overline{x}_1, \ldots, \overline{x}_m\} \cup \{C_1, \ldots, C_n\} \cup \\
& \{\theta^1_{i,j}, \ldots, \theta^{n+1}_{i,j} \mid i \in \{1, \ldots, m\}, j \in \{1, 2, 3\}\} \\
R^*_\mathcal{C} = {} & \{(x_i, \overline{x}_i), (\overline{x}_i, x_i) \mid i = 1, \ldots, m\} \cup \\
& \{(\dot{x}, C_j) \mid \dot{x} \in \{x, \overline{x}\}, \dot{x} \in C_j\} \cup \\
& \{(C_1, C_1), \ldots, (C_n, C_n)\} \cup \\
& \{(x_i, \theta^k_{i,1}), (\overline{x}_i, \theta^k_{i,2}), (\theta^k_{i,1}, \theta^k_{i,3}), (\theta^k_{i,2}, \theta^k_{i,3}), \\
& \quad (\theta^k_{i,3}, \theta^k_{i,3}) \mid i \in \{1, \ldots, m\}, k \in \{1, \ldots, n+1\}\}
\end{aligned}$$

An example of the above construction for $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5\}$ with $C_1 = \{x_1, \overline{x}_2\}$, $C_2 = \{\overline{x}_1, x_3\}$, $C_3 = \{x_2, \overline{x}_3\}$, $C_4 = \{x_2, x_3\}$, $C_5 = \{\overline{x}_1, x_2\}$ can be seen in Figure 2. We show that $l^* \in \mathbb{N}$ is the maximal number of jointly satisfiable clauses of $\mathcal{C}$ iff l^* is the maximal number with $\alpha \in \bigcup \mathsf{st}^*_{|R|-n+l^*}(F^*_\mathcal{C})$. For that it suffices to show that for any number l, if there are l clauses that can be jointly satisfied then $\alpha \in \bigcup \mathsf{st}^*_{|R|-n+l}(F^*_\mathcal{C})$, and vice versa. Before we are proving this, let us first discuss the motivation for introducing the θ-arguments. Since our aim is to provide a mapping between assignments for $\mathcal{C}$ and k-stable* extensions of $F^*_\mathcal{C}$ we must ensure that only one of x_i and $\overline{x}_i$ (for $i = 1, \ldots, m$) is contained in the considered k-stable* extensions. Since a k-stable* extension S is not necessarily conflict-free, it may happen that $x_i, \overline{x}_i \in S$. If this is the case, then necessarily

- $\theta^1_{i,1}, \ldots, \theta^{n+1}_{i,1} \in S$, or
- $\theta^1_{i,2}, \ldots, \theta^{n+1}_{i,2} \in S$, or
- $\theta^1_{i,3}, \ldots, \theta^{n+1}_{i,3} \in S$.

as well (or a combination of those), as S must have full range (so $\theta^1_{i,3}, \ldots, \theta^1_{i,3}$ must either be included or attacked by S). In any of these cases, the inclusion of these arguments amounts to $n + 1$ violated attacks. Without loss of generality, assume $x_i, \overline{x}_i \in S$ and $\theta^1_{i,3}, \ldots, \theta^{n+1}_{i,3} \in S$ (and $\theta^1_{i,1}, \ldots, \theta^{n+1}_{i,1} \notin S$ and $\theta^1_{i,2}, \ldots, \theta^{n+1}_{i,2} \notin S$). Also without loss of generality, assume $\overline{x}_i$ is the only argument in S that attacks $C_1, \ldots, C_l$ (for some $l \in \{1, \ldots, n\}$). Consider

$$\begin{aligned}
S' = {} & (S \setminus \{\overline{x}_i, \theta^1_{i,3}, \ldots, \theta^{n+1}_{i,3}\}) \cup \{\theta^1_{i,2}, \ldots, \theta^{n+1}_{1,2}\} \\
& \cup \{C_1, \ldots, C_l\}
\end{aligned}$$

Then S' is also a k-stable* extension of $F^*_\mathcal{C}$:

- $(S')^{\oplus}_{F^*_\mathcal{C}} = A^*_\mathcal{C}$: we removed $\overline{x}_i$ from S, but still $x_i \in S'$ which attacks $\overline{x}_i$. We removed $\theta^1_{i,3}, \ldots, \theta^{n+1}_{i,3}$ from S but added $\theta^1_{i,2}, \ldots, \theta^{n+1}_{1,2}$ which attack $\theta^1_{i,3}, \ldots, \theta^{n+1}_{i,3}$. Furthermore, since $\overline{x}_i$ has been removed, we added all arguments exclusively attacked by $\overline{x}_i$, namely $\theta^1_{i,2}, \ldots, \theta^{n+1}_{1,2}$ and $C_1, \ldots, C_l$. It follows that S' retains full range.
- $|(S')^{\circledast}_{F^*_\mathcal{C}}| \geq k$: by adding $C_1, \ldots, C_l$ to S' at most n attacks are now violated. But since we removed $\theta^1_{i,3}, \ldots, \theta^{n+1}_{i,3}$ and $\overline{x}_i$ the former $n + 3$ violated attacks are now satisfied. So if $|S^{\circledast}_{F^*_\mathcal{C}}| \geq k$ then $|(S')^{\circledast}_{F^*_\mathcal{C}}| \geq k$ as well.

The above shows that that if $F^*_\mathcal{C}$ has a k-stable* extension S, we can safely assume that either $x_i \in S$ or $\overline{x}_i \in S$ (but not both).

We now show both directions of the statement "there are l clauses that can be jointly satisfied iff $\alpha \in \bigcup \mathsf{st}^*_{|R|-n+l}(F^*_\mathcal{C})$":

- "$\Rightarrow$": Assume that l clauses can be jointly satisfied and, w.l.o.g., let these be $C_1, \ldots, C_l$. Let ω be a satisfying assignment of $C_1, \ldots, C_l$. Then define the set of arguments S via

$$\begin{aligned}
S = {} & \{\alpha\} \cup \\
& \{x_i, \theta^1_{i,2}, \ldots, \theta^{n+1}_{i,2} \mid \omega(x_i) = 1\} \cup \\
& \{\overline{x}_i, \theta^1_{i,1}, \ldots, \theta^{n+1}_{i,1} \mid \omega(x_i) = 0\} \cup \\
& \{C_{l+1}, \ldots, C_n\}
\end{aligned}$$

  We show that S is a $|R| - n + l$-stable* extension. First, note that S has full range: since $C_1, \ldots, C_l$ are satisfied by ω, there is $\dot{x}_i \in S$ with $\dot{x}_i \in C_j$ for some $i \in \{1, \ldots, m\}$, $\dot{x}_i \in \{x_i, \overline{x}_i\}$ and all $j = 1, \ldots, l$, so all of $C_1, \ldots, C_l$ are attacked by S. The remaining $C_{l+1}, \ldots, C_n$ are included in S. Furthermore, for each $i = 1, \ldots, m$,
  - either $x_i \in S$ or $\overline{x}_i \in S$ (and the other is attacked),
  - if $x_i \in S$ then $\theta^1_{i,1}, \ldots, \theta^{n+1}_{i,1}$ are attacked (and $\theta^1_{i,1}, \ldots, \theta^{n+1}_{i,1}$ are attacked if $\overline{x}_i \in S$, and
  - $\theta^1_{i,3}, \ldots, \theta^{n+1}_{i,3}$ are attacked by either $\theta^1_{i,1}, \ldots, \theta^{n+1}_{i,1}$ or $\theta^1_{i,2}, \ldots, \theta^{n+1}_{i,3}$.

  Finally, $\alpha \in S$. Note also that $|(S \times S) \cap R^*_\mathcal{C}| = n - l$ due to the self-attacks of $C_1, \ldots, C_l$ (no other attacks exist between elements of S). It follows $\alpha \in \bigcup \mathsf{st}^*_{|R|-n+l}(F^*_\mathcal{C})$.

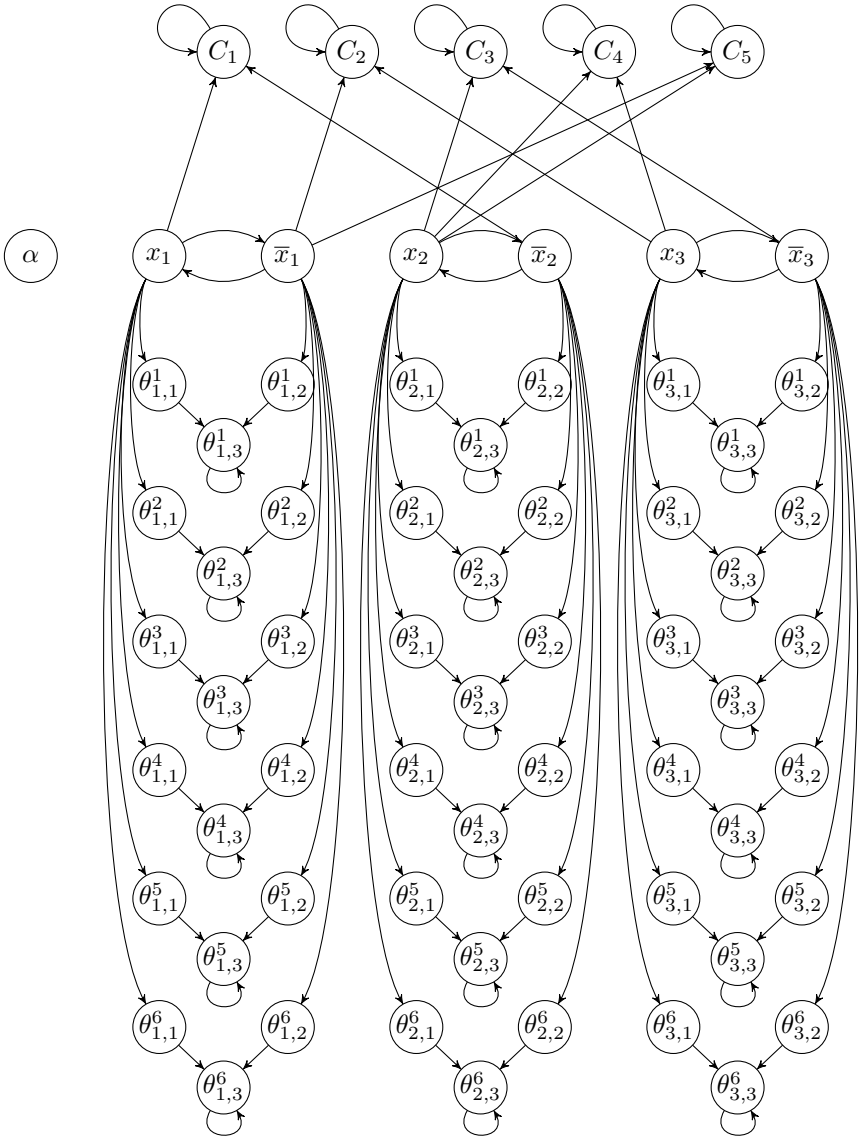


Figure 2: The AAF $F^*_{\mathcal{C}}$ for $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5\}$ with $C_1 = \{x_1, \overline{x}_2\}, C_2 = \{\overline{x}_1, x_3\}, C_3 = \{x_2, \overline{x}_3\}, C_4 = \{x_2, x_3\}, C_5 = \{\overline{x}_1, x_2\}$.

- "$\Leftarrow$": Assume $\alpha \in \bigcup \mathsf{st}^*_{|R|-n+l}(F^*_{\mathcal{C}})$, so let S be a $(|R| - n + l)$-stable* extension with $\alpha \in S$. As discussed above, we can assume that either $x_i \in S$ or $\overline{x}_i \in S$ (but not both) and that the only conflicts within S stem from some arguments of $\{C_1, \ldots, C_n\}$ being included in S. Without loss of generality, assume $C_{l+1}, \ldots, C_n \in S$. Similarly as in the proof of Theorem 4, we can now extract an assignment from S that satisfies clauses $C_1, \ldots, C_l$. □

Theorem 6. MAXSTABLE *is Poly-APX-complete under PTAS-reductions.*

Proof. Poly-APX-membership is witnessed by the following trivial algorithm. On input $F = (A, R)$ and $a \in A$, the algorithm either returns $-\infty$ in case $(a, a) \in A$ (for that case the algorithm satisfies $AR = 1$) or $\{a\}$ otherwise. In the latter case we have $APP \geq 1$ and therefore $AR \geq 1/|A|$ since $|A|$ is the maximal value of the optimal solution to MAXSTABLE. This is clearly a bound of the form $1/f(|A|)$ with a polynomial $f(n) = n$.

For Poly-APX-hardness, we provide a PTAS-reduction from MAXINDEPSET, which has been shown to be Poly-APX-hard under PTAS-reductions in [Bazgan *et al.*, 2005]. Let $\epsilon \in (0, 1)$ be an error parameter (which will be used to show the approximation quality below) and let $k = \lceil 1/\epsilon \rceil$.

On input $G = (V, E)$ define $F_{G,\epsilon} = (A_{G,\epsilon}, R_{G,\epsilon})$ via

$$A_{G,\epsilon} = V \cup \{b_1^v, \ldots, b_{k|V|}^v \mid v \in V\} \cup \{\alpha\}$$
$$R_{G,\epsilon} = \{(v, v'), (v', v) \mid \{v, v'\} \in E\} \cup \{(v, b_1^v), \ldots, (v, b_{k|V|}^v) \mid v \in V\} \cup \{(b_i^v, b_i^v) \mid v \in V, i = 1, \ldots, k|V|\}$$

An example of the above construction can be seen in Figure 3. First observe, that $S \subseteq A_{G,\epsilon}$ is a conflict-free set of $F_{G,\epsilon}$ if and only if $S \setminus \{\alpha\}$ is an independent set of G: no argument of the form b_i^v can be part of a conflict-free set of $F_{G,\epsilon}$ and two vertices are adjacent in G iff there is an attack between them in $F_{G,\epsilon}$. Also note that α can be part of any conflict-free set of $F_{G,\epsilon}$, since it has no incoming or outgoing attacks. Moreover, if S_1, S_2 are two independent sets with $|S_1| > |S_2|$ then $S_1 \cup \{\alpha\}, S_2 \cup \{\alpha\} \in A_{G,\epsilon}$ are two conflict-free sets in $F_{G,\epsilon}$ with $|(S_1)^{\oplus}_{F_{G,\epsilon}}| > |(S_2)^{\oplus}_{F_{G,\epsilon}}|$:

- Let $|V| = n$. Let S_1, S_2 be two independent sets with $|S_1| \geq |S_2|$. Note that each $v \in S_1$ contributes at least unique $kn + 1$ arguments to the value of $|(S_1)^{\oplus}_{F_G}|$ (the argument v itself and all arguments $b_1^v, \ldots, b_{kn}^v$). So we have

  $$|(S_1)^{\oplus}_{F_{G,\epsilon}}| \geq |S_1|(kn + 1)$$

  An upper bound for $|(S_2)^{\oplus}_{F_{G,\epsilon}}|$ can be obtained by assuming that $V \subseteq (S_2)^{\oplus}_{F_{G,\epsilon}}$, i. e., the range of S_2 contains (in addition to the $kn + 1$ unique arguments added by each argument in S_2) all arguments in V (which is an overestimation, since some arguments in V have already been accounted for in S_2). So we have

  $$|(S_2)^{\oplus}_{F_{G,\epsilon}}| \leq |S_2|(kn + 1) + n$$

  Due to $|S_1| > |S_2|$ we have

  $$\begin{aligned} |(S_1)^{\oplus}_{F_{G,\epsilon}}| &\geq |S_1|(kn + 1) \\ &\geq (|S_2| + 1)(kn + 1) \\ &= |S_2|(kn + 1) + (kn + 1) \\ &> |S_2|(kn + 1) + n \\ &\geq |(S_2)^{\oplus}_{F_{G,\epsilon}}| \end{aligned}$$

  and therefore the claim.

Moreover, if $S_1 \cup \{\alpha\}, S_2 \cup \{\alpha\} \in A_{G,\epsilon}$ are two conflict-free sets in $F_{G,\epsilon}$ with $|(S_1)^{\oplus}_{F_{G,\epsilon}}| > |(S_2)^{\oplus}_{F_{G,\epsilon}}|$ then $|S_1| \geq |S_2|$ follows analogously. This means, in particular, $S \cup \{\alpha\}$ is a solution to MAXSTABLE in $F_{G,\epsilon}$ (with query argument α) if and only if S is a solution to MAXINDEPSET in G. It should also be clear that $F_{G,\epsilon}$ can be computed in polynomial time (note, in particular, that ϵ is a constant).

What remains to be shown is that the reduction above fulfils the requirements of a PTAS-reduction for preserving the approximation quality. Let $S \subseteq A_{G,\epsilon}$ be an optimal solution and $S' \subseteq A_{G,\epsilon}$ be an approximate solution for MAXSTABLE in $F_{G,\epsilon}$. Remember that $k = \lceil 1/\epsilon \rceil$. Then we have

$$|S^{\oplus}_{F_{G,\epsilon}}| = (|S| - 1)(kn + 1) + m + 1$$
$$|(S')^{\oplus}_{F_{G,\epsilon}}| = (|S'| - 1)(kn + 1) + m' + 1$$

for some numbers $m, m' \in \{1, \ldots, n - 1\}$: m and m' are the numbers of arguments within V that are attacked by but not included in S and S', respectively; each argument in S/S' (except α) contributes $kn + 1$ individual arguments to the range; and α contributes 1 to the range. So the approximation ratio AR_1 for MAXSTABLE ist given by

$$AR_1 = \frac{(|S'| - 1)(kn + 1) + m' + 1}{(|S| - 1)(kn + 1) + m + 1}$$

Since $\alpha \in S$ and $\alpha \in S'$, the approximation ratio AR_2 for the corresponding MAXINDEPSET problem amounts to

$$AR_2 = \frac{|S'| - 1}{|S| - 1}$$

For a PTAS-reduction we have to show that there is a function $c : (0, 1) \to (0, 1)$ such that for all $\epsilon \in (0, 1)$ it holds that

$$AR_1 \geq 1 - c(\epsilon) \qquad \Longrightarrow \qquad AR_2 \geq 1 - \epsilon \tag{1}$$

Without loss of generality and to ease calculations, we assume that $k = 1/\epsilon$ is a natural number (otherwise we take the next smaller ϵ for which this is true). Moreover, we assume $|S'| \geq 2$, since $|S'| = 1$ implies $S' = \{\alpha\}$ and either $S = S'$ (in that case $AR_2 = 1$ and (1) is satisfied) or $|S| > 1$ and any non-self-attacking argument of $F_{G,\epsilon}$ could be added to S'. Given $|S'| \geq 2$ then having $|S| = 2$ also yields $AR_2 = 1$ and (1) is satisfied. So we assume $|S| \geq 3$.

We show (1) with c defined via

$$c(\epsilon) = \epsilon/2$$

for all $\epsilon \in (0, 1)$. Then it holds

$$\begin{aligned} & AR_1 \geq 1 - c(\epsilon) \\ \Leftrightarrow \quad & \frac{(|S'| - 1)(kn + 1) + m' + 1}{(|S| - 1)(kn + 1) + m + 1} \geq 1 - \epsilon/2 \\ \Leftrightarrow \quad & \frac{(|S'| - 1)(n/\epsilon + 1) + m' + 1}{(|S| - 1)(n/\epsilon + 1) + m + 1} \geq 1 - \epsilon/2 \\ \Rightarrow \quad & \frac{(|S'| - 1)(n/\epsilon + 1) + n}{(|S| - 1)(n/\epsilon + 1)} \geq 1 - \epsilon/2 \\ \Leftrightarrow \quad & (|S'| - 1)(n/\epsilon + 1) + n \geq (1 - \epsilon/2)(|S| - 1)(n/\epsilon + 1) \\ \Leftrightarrow \quad & (|S'| - 1) \geq (1 - \epsilon/2)(|S| - 1) - \frac{n}{n/\epsilon + 1} \\ \Leftrightarrow \quad & \frac{|S'| - 1}{|S| - 1} \geq 1 - \epsilon/2 - \frac{n}{(n/\epsilon + 1)(|S| - 1)} \\ \Leftrightarrow \quad & AR_2 \geq 1 - \epsilon + \underbrace{\epsilon/2 - \frac{n}{(n/\epsilon + 1)(|S| - 1)}}_{(*)} \end{aligned}$$

In order for (1) to hold it remains to show that the term (*) is

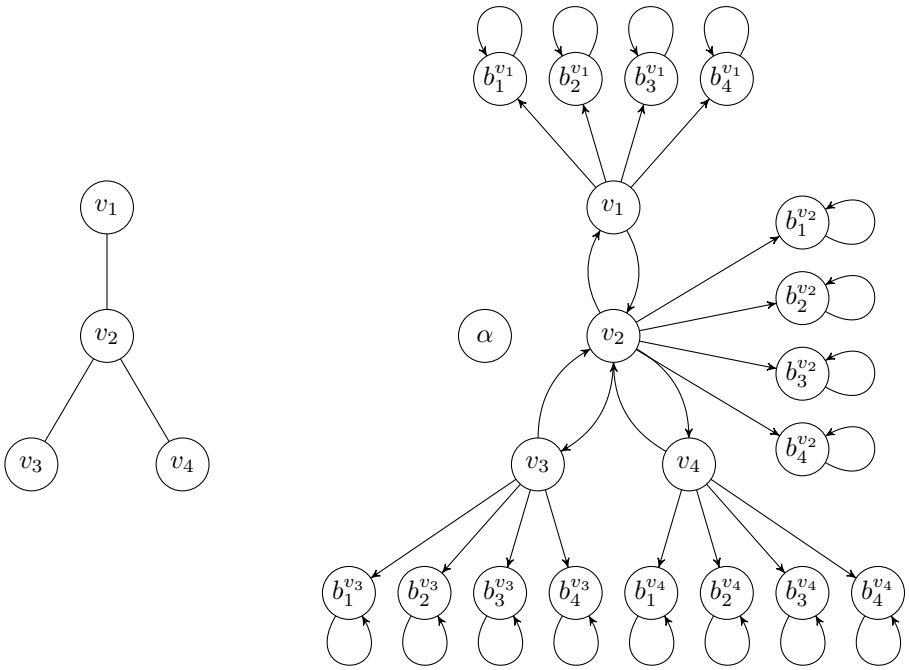


Figure 3: The reduction from the proof of Theorem 6; the graph $G = (V, E)$ on the left is translated to the AAF $F_{G,\epsilon}$ on the right (to keep the figure small we used $\epsilon = 1$, but note that for the construction we assume $\epsilon \in (0, 1)$).

non-negative:

$$
\begin{aligned}
& \epsilon/2 - \frac{n}{(n/\epsilon+1)(|S|-1)} \geq 0 \\
\Leftrightarrow\ & \epsilon/2 \geq \frac{n}{(n/\epsilon+1)(|S|-1)} \\
\Leftrightarrow\ & \epsilon/2(n/\epsilon+1)(|S|-1) \geq n \\
\Leftrightarrow\ & \frac{\epsilon n|S|}{2\epsilon} - \frac{\epsilon n}{2\epsilon} + \underbrace{\frac{\epsilon|S|}{2} - \frac{\epsilon}{2}}_{\geq 0} \geq n \\
\Leftarrow\ & \frac{\epsilon n|S|}{2\epsilon} - \frac{\epsilon n}{2\epsilon} \geq n \\
\Leftrightarrow\ & \frac{n|S|}{2} - \frac{n}{2} \geq n \\
\Leftrightarrow\ & \frac{|S|}{2} - \frac{1}{2} \geq 1 \\
\Leftrightarrow\ & |S| - 1 \geq 2 \\
\Leftrightarrow\ & |S| \geq 3
\end{aligned}
$$

which we assumed to be true. This shows (1) and concludes the proof. □

Theorem 7. MAXSTABLE* *is Log-APX-hard under PTAS-reductions.*

Proof. We provide a PTAS-reduction from MINDOMSET. Let $\epsilon \in (0, 1)$ be an error parameter (which will be used to show the approximation quality below) and let $k = \lceil 1/\epsilon \rceil$. On input $G = (V, E)$ define $F_{G,\epsilon} = (A_{G,\epsilon}, R_{G,\epsilon})$ via

$$A_{G,\epsilon} = V \cup \{b_1^v, \ldots, b_{k|V|^2}^v, c_1^v, \ldots, c_{k|V|^2}^v \mid v \in V\} \cup \{\alpha\}$$

$$
\begin{aligned}
R_{G,\epsilon} = \{&(v, v'), (v', v) \mid \{v, v'\} \in E\} \cup \\
&\{(v, b_1^v), \ldots, (v, b_{k|V|^2}^v) \mid v \in V\} \\
&\{(b_1^v, c_1^v), \ldots, (b_{k|V|^2}^v, c_{k|V|^2}^v) \mid v \in V\} \\
&\{(c_i^v, c_i^v) \mid v \in V, i = 1, \ldots, k|V|^2\}
\end{aligned}
$$

An example of the above construction can be seen in Figure 4.

We will first show that sets with full range in $A_{G,\epsilon}$ correspond to dominating sets in G:

1. If $S \subseteq A_{G,\epsilon}$ satisfies $S^{\oplus}_{F_{G,\epsilon}} = A_{G,\epsilon}$ then $S \cap V$ is a dominating set of G:
   Let $S \subseteq A_{G,\epsilon}$ satisfy $S^{\oplus}_{F_{G,\epsilon}} = A_{G,\epsilon}$. Note that arguments in $V \subseteq A_{G,\epsilon}$ are not attacked by any argument of the form b_i^v or c_i^v. So for every argument $v \in V$, either $v \in S$ or there is a $v' \in S$ with $v' R_{G,\epsilon} v$. Since $v' R_{G,\epsilon} v$ for $v', v \in V$ only holds if $\{v', v\} \in E$, we have that $S \cap V$ is a dominating set of G.
2. If $S \subseteq V$ is a dominating set of G then $\hat{S} = S \cup \{b_i^v \mid v \in V, i = 1, \ldots, n|V|^2\} \cup \{\alpha\}$ satisfies $\hat{S}^{\oplus}_{F_{G,\epsilon}} = A_{G,\epsilon}$:
   Let $S \subseteq V$ is a dominating set of G and $\hat{S} = S \cup \{b_i^v \mid v \in V, i = 1, \ldots, n|V|^2\} \cup \{\alpha\}$. So for every $v \in V$, either $v \in \hat{S}$ or there is $v' \in \hat{S}$ with $v' R_{G,\epsilon} v$. Furthermore $\alpha \in \hat{S}$ and all arguments of the form b_i^v are in $\hat{S}$ as well. Since every b_i^v attacks its c_i^v, all arguments of the latter form are also in the range of $\hat{S}$ It follows $\hat{S}^{\oplus}_{F_{G,\epsilon}} = A_{G,\epsilon}$.

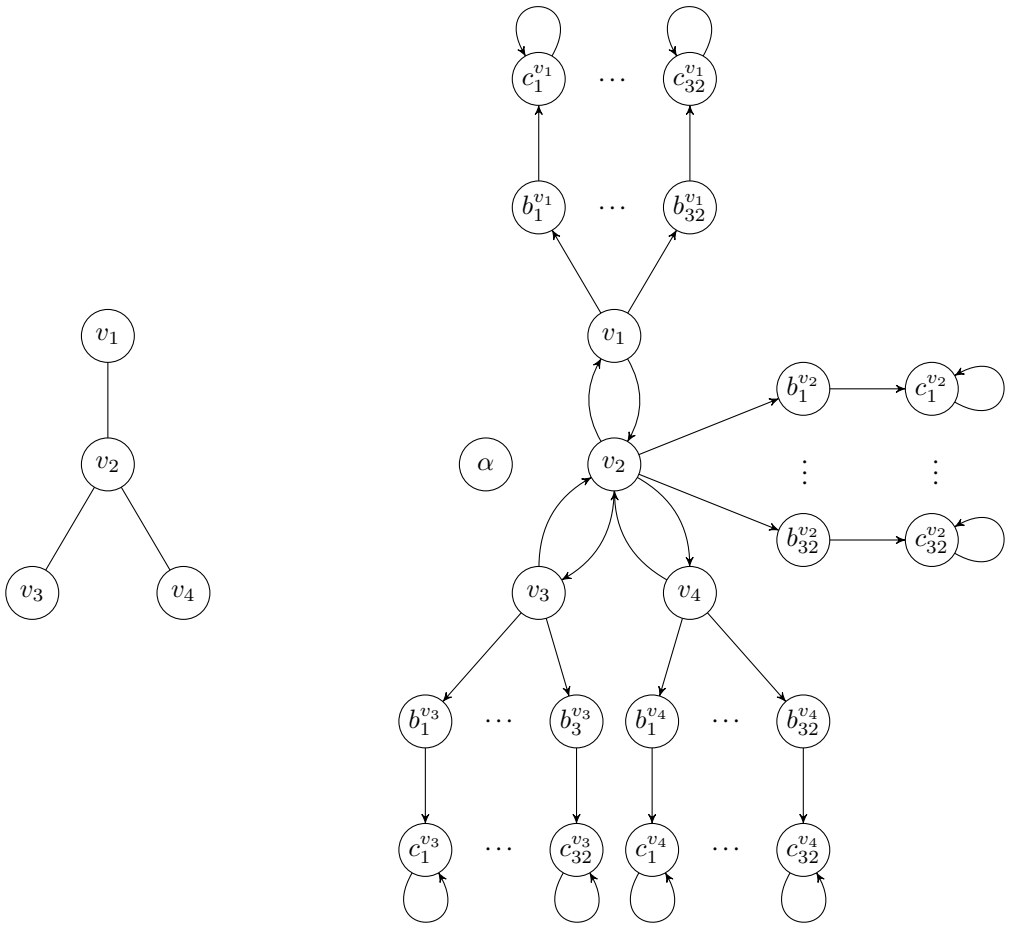


Figure 4: The reduction from the proof of Theorem 7; the graph $G = (V, E)$ on the left is translated to the AAF $F_{G,\epsilon}$ on the right (for $\epsilon = 1/2$ and therefore $k = 2$).

We now strengthen the above correspondence by showing that sets with full range that violate less attacks in $A_{G,\epsilon}$ correspond to smaller dominating sets in G. For a set $S \subseteq A_{G,\epsilon}$ recall that

$$S^{\circledast}_{F_{G,\epsilon}} = \{(a, b) \in R \mid a \notin S \text{ or } b \notin S\}$$

is the set of satisfied attacks of S in $F_{G,\epsilon}$. Let $S \subseteq A_{G,\epsilon}$ satisfy $S^{\oplus}_{F_{G,\epsilon}} = A_{G,\epsilon}$. We can assume

$$\{b_i^v \mid v \in V, i = 1, \ldots, n|V|^2\} \subseteq S$$
$$\{c_i^v \mid v \in V, i = 1, \ldots, n|V|^2\} \cap S = \emptyset$$

Since every c_i^v has to in the range of S, either $c_i^v \in S$ or $b_i^v \in S$. Assuming $b_i^v \in S$ and $c_i^v \notin S$ (for all v and i) will not decrease the value $|S^{\circledast}_{F_{G,\epsilon}}|$: having $c_i^v \in S$ will always contribute -1 to $|S^{\circledast}_{F_{G,\epsilon}}|$ (due to the self-attack) and having $b_i^v \in S$ instead, will *at most* contribute -1 to $|S^{\circledast}_{F_{G,\epsilon}}|$ (due to the attack from v to b_i^v). Let $|V| = n$, then we can write

$$|S^{\circledast}_{F_{G,\epsilon}}| = |R_{G,\epsilon}| - \text{viol}(S)$$

with

$$\text{viol}(S) = |S \cap V| kn^2 + m$$

and $m \in \{1, \ldots, 2|E|\}$. Here, $\text{viol}(S)$ is the number of violated attacks of S in $F_{G,\epsilon}$ and the number of satisfied attacks is the number of total attacks ($|R_{G,\epsilon}|$) minus the number of violated attacks. Furthermore, since we assume $\{b_i^v \mid v \in V, i = 1, \ldots, n|V|^2\} \subseteq S$, each $v \in S$ with $v \in V$ contributes kn^2 violated attacks (on the corresponding b_i^v), amounting to $|S \cap V| kn^2$ violated attacks concerning arguments of $S \cap V$ with arguments of the form b_i^v and c_i^v. Moreover, the term m captures the remaining violated attacks among the arguments in $S \cap V$ (and since every edge from G is duplicated, there are $2|E|$ many attacks between arguments in $S \cap V$). Since $|R_{G,\epsilon}|$ is fixed for a specific instance, the problem MAXSTABLE* can be equivalently defined by looking for sets S with full range that *minimise* $\text{viol}(S)$.

Now it holds

1. If $S, S' \subseteq A_{G,\epsilon}$ satisfy $S^{\oplus}_{F_{G,\epsilon}} = A_{G,\epsilon}$, $(S')^{\oplus}_{F_{G,\epsilon}} = A_{G,\epsilon}$ and $\text{viol}(S) < \text{viol}(S')$ then $|S \cap V| \leq |S' \cap V|$:

Let S, S' be as above. It holds

$$\begin{aligned} & \text{viol}(S) < \text{viol}(S') \\ \Leftrightarrow \quad & |S \cap V|kn^2 + m < |S' \cap V|kn^2 + m' \end{aligned}$$

for some $m, m' \in \{1, \ldots, 2|E|\}$. Note that $2|E| \leq n^2$. Then it holds

$$\begin{aligned} & |S \cap V|kn^2 + m < |S' \cap V|kn^2 + m' \\ \Rightarrow \quad & |S \cap V|kn^2 < |S' \cap V|kn^2 + kn^2 \\ \Leftrightarrow \quad & |S \cap V| < |S' \cap V| + 1 \\ \Leftrightarrow \quad & |S \cap V| \leq |S' \cap V| \end{aligned}$$

and therefore the claim.

2. If $S, S' \subseteq V$ are dominating sets of G with $|S| < |S'|$ then $\hat{S} = S \cup \{b_i^v \mid v \in V, i = 1, \ldots, n|V|^2\} \cup \{\alpha\}$ and $\hat{S}' = S \cup \{b_i^v \mid v \in V, i = 1, \ldots, n|V|^2\} \cup \{\alpha\}$ satisfy $\text{viol}(\hat{S}) < \text{viol}(\hat{S}')$:
Let $S, S' \subseteq V$ and $\hat{S}$, $\hat{S}'$ be as above. It holds

$$\begin{aligned} & |S| < |S'| \\ \Leftrightarrow \quad & |\hat{S} \cap V| < |\hat{S}' \cap V| \\ \Leftrightarrow \quad & |\hat{S} \cap V|kn^2 < |\hat{S}' \cap V|kn^2 \end{aligned}$$

Furthermore, due to $|S| < |S'|$ we have that $|\hat{S}' \cap V|kn^2 - |\hat{S} \cap V|kn^2 \geq kn^2 > m$ and therefore it follows

$$\begin{aligned} \Rightarrow \quad & |\hat{S} \cap V|kn^2 + m < |\hat{S}' \cap V|kn^2 \\ \Rightarrow \quad & |\hat{S} \cap V|kn^2 + m < |\hat{S}' \cap V|kn^2 + m' \\ \Leftrightarrow \quad & \text{viol}(\hat{S}) < \text{viol}(\hat{S}') \end{aligned}$$

and therefore the claim.

It follows that if S is an optimal solution for MAXSTABLE* in $F_{G,\epsilon}$ then $S \cap V$ is an optimal solution for MINDOMSET in G (and vice versa).

What remains to show is that the reduction above fulfils the requirements of a PTAS-reduction for preserving the approximation quality. Let $S \subseteq A_{G,\epsilon}$ be an optimal solution and $S' \subseteq A_{G,\epsilon}$ be an approximate solution for MAXSTABLE* in $F_{G,\epsilon}$. Then we have

$$AR_1 = \frac{|(S')^{\circledast}_{F_{G,\epsilon}}|}{|S^{\circledast}_{F_{G,\epsilon}}|}$$

as the approximation ratio for S' wrt. MAXSTABLE*. The approximation ratio AR_2 for the corresponding MINDOMSET instance is then

$$AR_2 = \frac{|S \cap V|}{|S' \cap V|}$$

For a PTAS-reduction we have to show that there is a function $c : (0,1) \to (0,1)$ such that for all $\epsilon \in (0,1)$ it holds that

$$AR_1 \geq 1 - c(\epsilon) \qquad \Longrightarrow \qquad AR_2 \geq 1 - \epsilon \qquad (2)$$

Without loss of generality and to ease calculations, we assume that $k = 1/\epsilon$ is a natural number (otherwise we take the next smaller ϵ for which this is true). We show (2) with c defined via

$$c(\epsilon) = \epsilon/2$$

for all $\epsilon \in (0,1)$. Then it holds

$$\begin{aligned} & AR_1 \geq 1 - c(\epsilon) \\ \Leftrightarrow \quad & \frac{|(S')^{\circledast}_{F_{G,\epsilon}}|}{|S^{\circledast}_{F_{G,\epsilon}}|} \geq 1 - \epsilon/2 \\ \Leftrightarrow \quad & \frac{|R_{G,\epsilon}| - \text{viol}(S')}{|R_{G,\epsilon}| - \text{viol}(S)} \geq 1 - \epsilon/2 \\ \Leftrightarrow \quad & \frac{|R_{G,\epsilon}| - |S' \cap V|kn^2 - m'}{|R_{G,\epsilon}| - |S \cap V|kn^2 - m} \geq 1 - \epsilon/2 \end{aligned}$$

Due to $m' \geq 0$ and $m \leq n^2$ we obtain

$$\begin{aligned} \Rightarrow \quad & \frac{|R_{G,\epsilon}| - |S' \cap V|kn^2}{|R_{G,\epsilon}| - |S \cap V|kn^2 - n^2} \geq 1 - \epsilon/2 \\ \Leftrightarrow \quad & |R_{G,\epsilon}| - |S' \cap V|kn^2 \\ & \quad \geq (1 - \epsilon/2)(|R_{G,\epsilon}| - |S \cap V|kn^2 - n^2) \\ \Leftrightarrow \quad & |R_{G,\epsilon}| - \frac{|S' \cap V|n^2}{\epsilon} \\ & \quad \geq (1 - \epsilon/2)(|R_{G,\epsilon}| - \frac{|S \cap V|n^2}{\epsilon} - n^2) \\ \Leftrightarrow \quad & |R_{G,\epsilon}| - \frac{|S' \cap V|n^2}{\epsilon} \geq |R_{G,\epsilon}| - \frac{\epsilon|R_{G,\epsilon}|}{2} \\ & \quad - \frac{|S \cap V|n^2}{\epsilon} + \frac{|S \cap V|n^2}{2} - n^2 + \frac{n^2\epsilon}{2} \\ \Leftrightarrow \quad & -\frac{|S' \cap V|n^2}{\epsilon} \geq -\frac{\epsilon|R_{G,\epsilon}|}{2} \\ & \quad - \frac{|S \cap V|n^2}{\epsilon} + \frac{|S \cap V|n^2}{2} - n^2 + \frac{n^2\epsilon}{2} \\ \Leftrightarrow \quad & -|S' \cap V| \geq \\ & \quad - \frac{\epsilon^2|R_{G,\epsilon}|}{2n^2} - |S \cap V| + \frac{\epsilon|S \cap V|}{2} - \epsilon + \frac{\epsilon^2}{2} \\ \Leftrightarrow \quad & 1 \leq \frac{\epsilon^2|R_{G,\epsilon}|}{2n^2|S' \cap V|} \\ & \quad + \frac{|S \cap V|}{|S' \cap V|} - \frac{\epsilon|S \cap V|}{2|S' \cap V|} + \frac{\epsilon}{|S' \cap V|} - \frac{\epsilon^2}{2|S' \cap V|} \\ \Leftrightarrow \quad & 1 - \frac{\epsilon^2|R_{G,\epsilon}|}{2n^2|S' \cap V|} - \frac{\epsilon}{|S' \cap V|} + \frac{\epsilon^2}{2|S' \cap V|} \\ & \quad \leq \frac{|S \cap V|}{|S' \cap V|} \underbrace{(1 - \epsilon/2)}_{\leq 1} \\ \Rightarrow \quad & 1 - \frac{\epsilon^2|R_{G,\epsilon}|}{2n^2|S' \cap V|} - \frac{\epsilon}{|S' \cap V|} + \frac{\epsilon^2}{2|S' \cap V|} \leq AR_2 \\ \Leftrightarrow \quad & AR_2 \geq 1 - \epsilon \\ & \quad \underbrace{+ \epsilon - \frac{\epsilon^2|R_{G,\epsilon}|}{2n^2|S' \cap V|} - \frac{\epsilon}{|S' \cap V|} + \frac{\epsilon^2}{2|S' \cap V|}}_{(*)} \end{aligned}$$

In order for (2) to hold it suffices to show that the term (*) is non-negative. Note that

$$|R_{G,\epsilon}| = 2|E| + 3kn = 2|E| + 3n/\epsilon$$

Then we have

$$\begin{aligned}
& \epsilon - \frac{\epsilon^2 |R_{G,\epsilon}|}{2n^2 |S' \cap V|} - \frac{\epsilon}{|S' \cap V|} + \frac{\epsilon^2}{2|S' \cap V|} \geq 0 \\
\Leftrightarrow \quad & \epsilon + \frac{\epsilon^2}{2|S' \cap V|} \geq \frac{2\epsilon^2 |E| + 3n\epsilon}{2n^2 |S' \cap V|} + \frac{\epsilon}{|S' \cap V|} \\
\Leftrightarrow \quad & 1 + \frac{\epsilon}{2|S' \cap V|} \geq \frac{2\epsilon |E| + 3n}{2n^2 |S' \cap V|} + \frac{1}{|S' \cap V|} \\
\Leftrightarrow \quad & 2|S' \cap V| + \epsilon \geq \frac{2\epsilon |E|}{n^2} + \frac{3}{n} + 2
\end{aligned}$$

We now assume $n \geq 2$ (note that problems with $n = 1$ are trivial, since the set containing the single vertex is the only dominating set). So we have $3/n \leq 3/2$ and also due to $2|E| \leq n^2$ and $3/n \leq 3$ we have

$$\begin{aligned}
\Leftarrow \quad & 2|S' \cap V| + \epsilon \geq \frac{\epsilon n^2}{n^2} + 3/2 + 2 \\
\Leftrightarrow \quad & 2|S' \cap V| + \epsilon \geq \epsilon + 3\frac{1}{2} \\
\Leftrightarrow \quad & 2|S' \cap V| \geq 3\frac{1}{2} \\
\Leftrightarrow \quad & |S' \cap V| \geq 1\frac{3}{4}
\end{aligned}$$

So (2) holds for $|S' \cap V| \geq 2$. The remaining case $|S' \cap V| = 1$ is trivial, since then $|S \cap V| = 1$ as well and $AR_2 = 1$. This concludes the proof. □

Proposition 4. *Let k be the output of the GCF algorithm on input $F = (A, R)$ and $a \in A$. Let furthermore $k^\dagger$ be the optimal solution to* MAXSTABLE *on F and a. Then*

1. *there is a conflict-free set S with $a \in S$ and $k = |S_F^{\oplus}|$,*
2. *$k \geq k^\dagger / |A|$, and*
3. *k has been determined by GCF in polynomial time.*

Proof.

1. Line 6 of Algorithm 1 only adds arguments b to S auch that $S \cup \{b\}$ is conflict-free. Moreover, $a \in S$ due to line 3 and the output k of Algorithm 1 is exactly $|S_F^{\oplus}|$ (line 7).
2. If Algorithm 1 terminates in line 2 then $k = k^\dagger = -\infty$ and $-\infty \geq -\infty/|A|$ is trivially satisfied. Otherwise, $|S| \geq 1$ due to line 3 and therefore $k = |S_F^{\oplus}| \geq 1$ as well. Since $k^\dagger \leq |A|$, the claim also follows in that case.
3. Note that the number of iterations of the **while**-loop in lines 4–6 is bounded by $|A|$ (as each argument can only be added once to S. Determining b in line 5 can be accomplished by iterating through all candidates $b \in A$, checking $b \in A \setminus (S \cup S_F^+ \cup S_F^-)$, determining $(S \cup \{b\})_F^{\oplus}$, and selecting a b with maximal value. All this can also be done in polynomial time. □

Proposition 5. *Let k be the output of the SCF algorithm on input $F = (A, R)$ and $a \in A$. Let furthermore $k^\dagger$ be the optimal solution to* MAXSTABLE *on F and a. Then*

1. *there is a conflict-free set S with $a \in S$ and $k = |S_F^{\oplus}|$,*
2. *$k \geq k^\dagger / |A|$, and*
3. *k has been determined by SCF in polynomial time.*

Proof.

1. Line 4 of Algorithm 2 ensures that the set S is conflict-free upon termination. Moreover, $a \in S$ due to line 3 and the fact that a is never removed in line 5. Finally, the output k of Algorithm 2 is exactly $|S_F^{\oplus}|$ (line 7).
2. If Algorithm 2 terminates in line 2 then $k = k^\dagger = -\infty$ and $-\infty \geq -\infty/|A|$ is trivially satisfied. Otherwise, $|S| \geq 1$ as argument a is never removed from S and $\{a\}$ is conflict-free and thus satisfies the condition line line 4, therefore $k = |S_F^{\oplus}| \geq 1$ as well. Since $k^\dagger \leq |A|$, the claim also follows in that case.
3. Note that the number of iterations of the **while**-loop in lines 4–6 is bounded by $|A|$ (as each argument can only be removed once to S. Determining b in line 5 can be accomplished by iterating through all candidates $b \in S$ and determining $(S \setminus \{b\})_F^{\circledast}$, and selecting a b with maximal value. All this can also be done in polynomial time. □

Proposition 6. *Let k be the output of the GFR algorithm on input $F = (A, R)$ and $a \in A$. Let furthermore $k^\dagger$ be the optimal solution to* MAXSTABLE* *on F and a. Then*

1. *there is a set S with $a \in S$, $S_F^{\oplus} = A$, and $k = |S_F^{\circledast}|$,*
2. *$k \geq k^\dagger / |R|$, and*
3. *k has been determined by GFR in polynomial time.*

Proof.

1. Line 1 ensures $a \in S$. Line 2 ensures that upon termination $S_F^{\oplus} = A$. The output $k = |S_F^{\circledast}|$ is defined as such in line 5.
2. Note first that $k^\dagger \leq |R|$. If $k^\dagger = 0$ then the statement holds trivially, so assume $k^\dagger \geq 1$. Then it suffices to show $k = |(S_{\text{output}})_F^{\circledast}| \geq 1$ for the set S_{output} determined by GFR. Assume $|(S_{\text{output}})_F^{\circledast}| = 0$. As $(S_{\text{output}})_F^{\oplus} = A$ it follows $S_{\text{output}} = A$: if there would be $b \in A \setminus S_{\text{output}}$ then there would be $c \in S_{\text{output}}$ and cRb, therefore having $|(S_{\text{output}})_F^{\circledast}| \geq 1$. So let $S_{\text{output}} = \{a_0, a_1, \ldots, a_n\} = A$ with $a_0 = a$ and let $a_1, \ldots, a_n$ be also the order the arguments are added to S in line 4. Let x_i be the corresponding values $|(S \cup \{a_i\})_F^{\oplus}|$ of each a_i from line 3 (so x_i is the number of new arguments brought into the range of S in that iteration, and $x_0 = |\{a\}_F^{\oplus}|$), $i = 0, \ldots, n$. Then we have $x_0 + x_1 + \ldots + x_n = |A|$ and also $x_i \geq 1$ for each $i = 1, \ldots, n$ (since a_i brings at least itself into the range, otherwise there is no need to pick a_i). Moreover, we have

$$x_1 \geq x_2 \geq \ldots \geq x_n$$

as arguments are picked according to the number of arguments they bring into the range. So it follows

$$x_0 = x_1 = x_2 = \ldots = x_n = 1$$

In particular $x_1 = 1$, meaning that the maximal range *every* argument has is 1 (so just itself). It follows that the optimal solution must also contain all arguments and we have $k = k^\dagger$. So we have $k \geq k^\dagger/|R|$ in general.

3. The **while**-loop in lines 2–4 has at most $|A|$ iterations, since each argument can only be added once to S. Determining $|(S \cup \{b\})^\oplus_F|$ in line 3 can also be done in polynomial time. □

Proposition 7. *Let k be the output of the SFR algorithm on input $F = (A, R)$ and $a \in A$. Let furthermore $k^\dagger$ be the optimal solution to* MAXSTABLE* *on F and a. Then*

1. *there is a set S with $a \in S$, $S^\oplus_F = A$, and $k = |S^\circledast_F|$,*
2. *$k \geq k^\dagger/|R|$, and*
3. *k has been determined by SFR in polynomial time.*

Proof.

1. Line 1 and the selection of $b \in S \setminus \{a\}$ ensures that $a \in S$ upon termination. Since $A^\oplus_F = A$ we have $S^\oplus_F = A$ before line 2. Line 4 is only executed when the selected b is not jeopardising full range of S, so $S^\oplus_F = A$ upon termination as well. The output $k = |S^\circledast_F|$ is defined as such in line 5.
2. Note first that $k^\dagger \leq |R|$. If $k^\dagger = 0$ then the statement holds trivially, so assume $k^\dagger \geq 1$. Then it suffices to show $k = |(S_{\text{output}})^\circledast_F| \geq 1$ for the set S_{output} determined by SFR. Since $k^\dagger \geq 1$, there is at least one argument $b \in A$ that is not contained in an optimal solution and involved in at least one attack. Then b also satisfies the condition of line 2 (in the first iteration of the **while**-loop) and yields $|(S \setminus \{b\})^\circledast_F| \geq 1$. So whichever argument b' is selected in the first iteration of the **while**-loop also has $|(S \setminus \{b'\})^\circledast_F| \geq 1$. As b' is removed from S, the attacks related to b' will remain satisfied and we have $|(S_{\text{output}})^\circledast_F| \geq 1$.
3. The **while**-loop in lines 2–4 has at most $|A|$ iterations, since each argument can only be removed once from S. Determining b in line 3 (and implicitly checking the condition in line 2) can be done by considering each candidate $b \in A$ and checking $(S \setminus \{b\})^\oplus_F = A$ and determining $|(S \setminus \{b\})^\circledast_F|$, each of these operations can be done in polynomial time. □

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