Characterising Serialisation Equivalence for Abstract Argumentation

Proofs of technical results

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Proposition 1. *Let F and G be argumentation frameworks. It holds that $F^{gk} = G^{gk}$ if and only if $F \equiv^{s}_{\mathsf{sa}} G$.*

Proof. This follows easily from the proofs of Lemma 6 and Theorem 3 from [3]. The proof of Lemma 6 in [3] shows that $\mathsf{gr}(F) = \mathsf{gr}(F^{gk})$ by demonstrating that the characteristic functions Γ_F and $\Gamma_{F^{gk}}$ for F and F^{gk} return the same set of defended arguments for all conflict-free input sets S. In particular, that means we have $\Gamma_F(\emptyset) = \Gamma_{F^{gk}}(\emptyset)$. Furthermore, it also holds that $\Gamma_F(S_i) = \Gamma_{F^{gk}}(S_i)$ with $S_0 = \emptyset$ and $S_i = \Gamma_F(S_{i-1})$ for all $i > 0$. From the proof of the above statement, it follows directly that the same holds for any $S' \subseteq S_i$, i. e., $\Gamma_F(S') = \Gamma_{F^{gk}}(S')$. Clearly, this implies that the strongly admissible extensions are also the same for F and F^{gk}. The proof of Theorem 3 in [3] can then directly be applied to the sa semantics. It follows that strong equivalence wrt. strongly admissible semantics is characterised by the grounded kernel. □

Theorem 3. *Let F, G be argumentation frameworks. Then it holds that*

1. *$F \equiv^{se}_{\mathsf{ad}} G$ if and only if $F \equiv^{se}_{\mathsf{pr}} G$,*
2. *If $F \equiv^{se}_{\mathsf{ad}} G$, then $F \equiv^{se}_{\mathsf{uc}} G$,*
3. *If $F \equiv^{se}_{\mathsf{pr}} G$, then $F \equiv^{se}_{\mathsf{uc}} G$,*
4. *If $F \equiv^{se}_{\mathsf{co}} G$, then $F \equiv^{se}_{\mathsf{pr}} G$,*
5. *If $F \equiv^{se}_{\mathsf{co}} G$, then $F \equiv^{se}_{\mathsf{ad}} G$,*
6. *If $F \equiv^{se}_{\mathsf{co}} G$, then $F \equiv^{se}_{\mathsf{uc}} G$,*
7. *$F \equiv^{se}_{\mathsf{sa}} G$ if and only if $F \equiv^{se}_{\mathsf{gr}} G$.*

Proof. We proof each statement individually as follows:

- *$F \equiv^{se}_{\mathsf{ad}} G$ if and only if $F \equiv^{se}_{\mathsf{pr}} G$.*
  We show both directions of the above statement as follows.

  "⇒" It follows from Theorem 1, that $\mathfrak{S}_{\mathsf{pr}}(F)$ is the subset of $\mathfrak{S}_{\mathsf{ad}}(F)$, containing exactly those sequences $(S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{ad}}(F)$ where $\mathsf{IS}(F^{S_1 \cup \dots \cup S_n}) = \emptyset$. Taking Definition 8 into account, it follows from $F \equiv^{se}_{\mathsf{ad}} G$ that $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G)$. This results in $\mathfrak{S}_{\mathsf{pr}}(F) \subseteq \mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G) \supseteq \mathfrak{S}_{\mathsf{pr}}(G)$.

  Let us assume $\mathfrak{S}_{\mathsf{pr}}(F) \neq \mathfrak{S}_{\mathsf{pr}}(G)$. This would mean, w.l.o.g., there exists some serialisation sequence $\mathcal{S} = (S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{pr}}(F)$ with $\mathcal{S} \notin \mathfrak{S}_{\mathsf{pr}}(G)$. Then, one of the following two cases must apply:

  (1) $\mathcal{S}$ is not an admissible serialisation sequence in G.
  That however contradicts our initial assumption of $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G)$.

  (2) $\mathcal{S}$ is not a maximal serialisation sequence in G,i. e., we have that $\mathsf{IS}(G^{S_1 \cup \dots \cup S_n}) \neq \emptyset$. Then, there is some $S' \in \mathsf{IS}(G^{S_1 \cup \dots \cup S_n})$ such that $(S_1, \dots, S_n, S')$ is an admissible serialisation sequence of G. Clearly, because of $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G)$ it follows that $(S_1, \dots, S_n, S')$ is an admissible serialisation sequence of F and thus $(S_1, \dots, S_n)$ cannot be a preferred serialisation sequence of F contradicting our initial assumption.

  Therefore, it always holds that $\mathfrak{S}_{\mathsf{pr}}(F) = \mathfrak{S}_{\mathsf{pr}}(G)$ and hence $F \equiv^{se}_{\mathsf{ad}} G$ implies $F \equiv^{se}_{\mathsf{pr}} G$ for all AFs.

  "⇐" If $F \equiv^{se}_{\mathsf{pr}} G$ it follows from Definition 8 that $\mathfrak{S}_{\mathsf{pr}}(F) = \mathfrak{S}_{\mathsf{pr}}(G)$. We now show that for some arbitrary serialisation sequence $\mathcal{S} = (S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{ad}}(F)$ it follows that $\mathcal{S} \in \mathfrak{S}_{\mathsf{ad}}(G)$. From $\mathcal{S} = (S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{ad}}(F)$ it follows that either

  (1) we have that $\mathcal{S} \in \mathfrak{S}_{\mathsf{pr}}(F)$, or

  (2) there exists some preferred serialisation sequence $\mathcal{S}' = (S_1, \dots, S_n, T_1 \dots, T_m) \in \mathfrak{S}_{\mathsf{pr}}(F)$.

  For (1), it follows directly that $\mathcal{S} \in \mathfrak{S}_{\mathsf{pr}}(G)$ and thus $\mathcal{S} \in \mathfrak{S}_{\mathsf{ad}}(G)$.

  For (2), consider the following statement which follows directly from Theorem 1. It holds that $\mathfrak{S}_{\mathsf{ad}}(F) = \{(S_1, \dots, S_i) \mid 1 \leq i \leq n \wedge (S_1, \dots, S_n) \in \mathfrak{S}_{\mathsf{pr}}(F)\}$, i. e., every subsequence of a preferred serialisation sequence is an admissible serialisation sequence. Clearly, from $\mathcal{S}' \in \mathfrak{S}_{\mathsf{pr}}(F)$ it follows that $\mathcal{S}' \in \mathfrak{S}_{\mathsf{pr}}(G)$. Together with the above statement and since $\mathcal{S}$ is a sub-sequence of $\mathcal{S}'$, it follows that $\mathcal{S} \in \mathfrak{S}_{\mathsf{ad}}(G)$.

  Therefore, $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G)$ has to be true and it follows that $F \equiv^{se}_{\mathsf{pr}} G$ implies $F \equiv^{se}_{\mathsf{ad}} G$ for all AFs.

- *If $F \equiv^{se}_{\mathsf{ad}} G$, then $F \equiv^{se}_{\mathsf{uc}} G$.*
  Consider any two argumentation frameworks $F = (\mathcal{A}_F, \mathcal{R}_F)$ and $G = (\mathcal{A}_G, \mathcal{R}_G)$ with $F \equiv^{se}_{ad} G$. Then we have that $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G)$ and due to Proposition 2 also $\mathsf{ad}(F) = \mathsf{ad}(G)$. We show now, w.l.o.g, that for any serialisation sequence $\mathcal{S} = (S_1, \dots S_n)$ if $\mathcal{S} \in \mathfrak{S}_{\mathsf{uc}}(F)$ then $\mathcal{S} \in \mathfrak{S}_{\mathsf{uc}}(G)$. We proof inductively over the length of the sequence $(S_1, \dots S_n)$ for $i = 1, \dots, n$.

  - Consider the base case i=1. From $\mathsf{ad}(F) = \mathsf{ad}(G)$ it directly follows that $\mathsf{IS}(F) = \mathsf{IS}(G)$. Per definition of the unchallenged semantics, we have that $S_1 \in \mathsf{IS}^{\not\leftarrow}(F) \cup \mathsf{IS}^{\not\leftrightarrow}(F)$. Assume the contrary, i. e., $S_1 \notin \mathsf{IS}^{\not\leftarrow}(G) \cup \mathsf{IS}^{\not\leftrightarrow}(G)$. Then we must have $S_1 \in \mathsf{IS}^{\leftrightarrow}(G)$ because of $\mathsf{IS}(F) = \mathsf{IS}(G)$. That means, there exists some $S' \in \mathsf{IS}(G)$ such that $S'\mathcal{R}_G S_1$. Since both S_1 and S' are admissible, it follows that S_1 defends itself against S' in G, i.e., we have $S_1\mathcal{R}_G S'$ and thus $S' \in \mathsf{IS}^{\leftrightarrow}(G)$. Because of $\mathsf{IS}(F) = \mathsf{IS}(G)$, it follows that $S' \in \mathsf{IS}(F)$. Since $S_1 \notin \mathsf{IS}^{\leftrightarrow}(F)$ it also follows that $S' \not\mathcal{R}\ S_1$. Consider now the sequence (S', S_1). We have that $S' \in \mathsf{IS}(F)$ and because of $S' \not\mathcal{R}_F S_1$ it also holds that $S_1 \in \mathsf{IS}(F^{S'})$. Thus, (S', S_1) is an admissible serialisation sequence of F, i.e., $(S', S_1) \in \mathfrak{S}_{\mathsf{ad}}(F)$. That means we must have $(S', S_1) \in \mathfrak{S}_{\mathsf{ad}(G)}$. However, we know that $S'\mathcal{R}_G S_1$ which leads to $S_1 \notin \mathsf{IS}(G^{S'})$ and thus a contradiction because of $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G)$.

– Now consider some arbitrary i with $2 \leq i \leq n$. It holds that $S_i \in \mathsf{IS}^{\not\leftarrow}(F^{S_1\cup\dots\cup S_{i-1}}) \cup \mathsf{IS}^{\not\leftrightarrow}(F^{S_1\cup\dots\cup S_{i-1}})$. Assume this does not hold for G, i.e., we have $S_i \in \mathsf{IS}^{\leftrightarrow}(G^{S_1\cup\dots\cup S_{i-1}})$. Like in the base case, it follows there exists some $S' \in \mathsf{IS}^{\leftrightarrow}(G^{S_1\cup\dots\cup S_{i-1}})$ which attacks S_1 in $G^{S_1\cup\dots\cup S_{i-1}}$. Then, one of the following two cases must apply: (1) $S' \in \mathsf{IS}(F^{S_1\cup\dots\cup S_{i-1}})$, or (2) $S' \notin \mathsf{IS}(F^{S_1\cup\dots\cup S_{i-1}})$.

For (1), the contradiction can be shown like in the base case.

For (2), consider the sequence $(S_1, \dots, S_{i-1}, S')$. Clearly, it holds that $S_j \in \mathsf{IS}(G^{S_1\cup\dots\cup S_j})$ for all j with $1 \leq j < i$ and also that $S' \in \mathsf{IS}(G^{S_1\cup\dots\cup S_{i-1}})$. Thus, $(S_1, \dots, S_{i-1}, S')$ is an admissible serialisation sequence of G. However, because of $S' \notin \mathsf{IS}(F^{S_1\cup\dots\cup S_{i-1}})$ it follows that $(S_1, \dots, S_{i-1}, S') \notin \mathfrak{S}_{\mathsf{ad}}(F)$ which is a contradiction.

All in all, it follows that if $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G)$ then $\mathfrak{S}_{\mathsf{uc}}(F) = \mathfrak{S}_{\mathsf{uc}}(G)$.

- *If $F \equiv^{se}_{pr} G$, then $F \equiv^{se}_{uc} G$.*
  Follows directly from the two proofs above, i.e., serialisation equivalence wrt. admissible and preferred semantics coincide and serialisation equivalence wrt. admissible semantics implies serialisation equivalence wrt. unchallenged semantics.
- *If $F \equiv^{se}_{co} G$, then $F \equiv^{se}_{pr} G$.*
  Consider any two argumentation frameworks F, G with $F \equiv^{se}_{\mathsf{co}} G$. It follows from Theorem 1, that $\mathfrak{S}_{\mathsf{pr}}(F)$ is a subset of $\mathfrak{S}_{\mathsf{co}}(F)$, containing all sequences $\mathcal{S} = (S_1, \dots S_n)$ with $\mathsf{IS}(F^{S_1\cup\dots\cup S_n}) = \emptyset$. Taking Definition 8 into account, it follows from $F \equiv^{se}_{\mathsf{co}} G$ that $\mathfrak{S}_{\mathsf{co}}(F) = \mathfrak{S}_{\mathsf{co}}(G)$. This results in $\mathfrak{S}_{\mathsf{pr}}(F) \subseteq \mathfrak{S}_{\mathsf{co}}(F) = \mathfrak{S}_{\mathsf{co}}(G) \supseteq \mathfrak{S}_{\mathsf{pr}}(G)$.
  Let us assume $\mathfrak{S}_{\mathsf{pr}}(F) \neq \mathfrak{S}_{\mathsf{pr}}(G)$. This means, w.l.o.g., there exists some serialisation sequence $\mathcal{S} = (S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{pr}}(F)$ with $\mathcal{S} \notin \mathfrak{S}_{\mathsf{pr}}(G)$. That implies there is some serialisation sequence $\mathcal{S}' \in \mathfrak{S}_{\mathsf{pr}}(G)$ such that $\mathcal{S}$ is a sub-sequence of $\mathcal{S}'$. However, that also implies $\mathcal{S}' \in \mathfrak{S}_{\mathsf{co}}(G)$ and thus $\mathcal{S}' \in \mathfrak{S}_{\mathsf{co}}(F)$. That contradicts our assumption of $\mathcal{S} \in \mathfrak{S}_{\mathsf{pr}}(F)$. Therefore, it holds that $\mathfrak{S}_{\mathsf{pr}}(F) = \mathfrak{S}_{\mathsf{pr}}(G)$ and hence $F \equiv^{se}_{\mathsf{pr}} G$.
- *If $F \equiv^{se}_{co} G$, then $F \equiv^{se}_{ad} G$.*
  Follows directly from the fact that serialisation equivalence wrt. admissible and preferred semantics coincide and that serialisation equivalence wrt. complete semantics implies serialisation equivalence wrt. preferred semantics.
- *If $F \equiv^{se}_{co} G$, then $F \equiv^{se}_{uc} G$.*
  Follows directly via transitivity, i.e., serialisation equivalence wrt. complete semantics implies serialisation equivalence wrt. admissible semantics which in turn implies serialisation equivalence wrt. unchallenged semantics.
- *$F \equiv^{se}_{sa} G$ if and only if $F \equiv^{se}_{gr} G$.*
  We show both directions of the above statement as follows.

  "⇒" It follows from Theorem 1, that $\mathfrak{S}_{\mathsf{gr}}(F)$ is a subset of $\mathfrak{S}_{\mathsf{sa}}(F)$, containing all sequences corresponding to the subset-maximal extensions. Taking Definition 8 into account, it follows from $F \equiv^{se}_{\mathsf{sa}} G$ that $\mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(G)$. This results in $\mathfrak{S}_{\mathsf{gr}}(F) \subseteq \mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(G) \supseteq \mathfrak{S}_{\mathsf{gr}}(G)$.

  Let us assume $\mathfrak{S}_{\mathsf{gr}}(F) \neq \mathfrak{S}_{\mathsf{gr}}(G)$. Then, w.l.o.g., there must be some serialisation sequence $\mathcal{S} = (S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{gr}}(F)$ with $\mathcal{S} \notin \mathfrak{S}_{\mathsf{gr}}(G)$. From $\mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(G)$ it follows that F and G have the same strongly admissible extensions and thus also the same grounded extension E_{gr} which means $S_1 \cup \dots \cup S_n = E_{\mathsf{gr}}$. Due to $\mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(G)$ we also know that $\mathcal{S} \in \mathfrak{S}_{\mathsf{sa}}(G)$. Thus, $\mathcal{S}$ must also be a grounded serialisation sequence in G.

  Therefore, it holds that $\mathfrak{S}_{\mathsf{gr}}(F) = \mathfrak{S}_{\mathsf{gr}}(G)$ and hence $\mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(G)$ implies $F \equiv^{se}_{\mathsf{gr}} G$ for all AFs.

  "⇐" Consider the following statement which follows directly from Theorem 1. It holds that $\mathfrak{S}_{\mathsf{sa}}(F) = \{(S_1, \dots, S_i) \mid i \leq n \wedge (S_1, \dots, S_n) \in \mathfrak{S}_{\mathsf{gr}}(F)\}$, i.e., every sub-sequence of a grounded serialisation sequence is a strongly admissible serialisation sequence. If $F \equiv^{se}_{\mathsf{gr}} G$ it follows with Definition 8 that $\mathfrak{S}_{\mathsf{gr}}(F) = \mathfrak{S}_{\mathsf{gr}}(G)$.

  Let us assume that $\mathfrak{S}_{\mathsf{sa}}(F) \neq \mathfrak{S}_{\mathsf{sa}}(G)$. In this case, w.l.o.g., there exists some serialisation sequence $\mathcal{S} = (S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{sa}}(F)$ with $\mathcal{S} \notin \mathfrak{S}_{\mathsf{sa}}(G)$. Then there must exist a serialisation sequence $\mathcal{S}' = (S_1, \dots, S_n, \dots, S_m) \in \mathfrak{S}_{\mathsf{gr}}(F)$ which is maximal, i.e., $\mathsf{IS}^{\not\leftarrow}(F^{S_1\cup\dots\cup S_n\cup\dots\cup S_m}) = \emptyset$. Since $\mathfrak{S}_{\mathsf{gr}}(F) = \mathfrak{S}_{\mathsf{gr}}(G)$ it follows that $\mathcal{S}' \in \mathfrak{S}_{\mathsf{gr}}(G)$. Together with the previously stated observation, it follows that $\mathcal{S}$ must be in $\mathfrak{S}_{\mathsf{sa}}(G)$. Therefore, $\mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(G)$ must hold and it follows that $F \equiv^{se}_{\mathsf{gr}} G$ implies $F \equiv^{se}_{\mathsf{sa}} G$ for all AFs. ☐

Theorem 4. *Let $\sigma \in \Sigma$ be a semantics. For any two argumentation frameworks F and G, if $F \equiv^{se}_{\sigma} G$, then it follows that $F \equiv_{\sigma} G$.*

Proof. Let σ be a serialisable semantics. We will show that if two argumentation frameworks F and G are serialisation equivalent wrt. σ, then they are also standard equivalent wrt. σ, i.e., it follows from $F \equiv^{se}_{\sigma} G$, that $F \equiv_{\sigma} G$.

Assume that $F \equiv^{se}_{\sigma} G$. From Definition 8 it follows directly that $\mathfrak{S}_{\sigma}(F) = \mathfrak{S}_{\sigma}(G)$. Now, assume the theorem does not hold and we have that $F \not\equiv_{\sigma} G$. Per Definition 6, that means we have $\sigma(F) \neq \sigma(G)$. Thus, there must exist some $E \in \sigma(F)$ with $E \notin \sigma(G)$, or vice versa. However, since σ is serialisable, it follows from Theorem 1 that for any serialisable semantics σ, if we have that $E \in \sigma(F)$ then there must exist a serialisation sequence $(S_1, \dots, S_n) \in \mathfrak{S}_{\sigma}(F)$ with $E = S_1 \cup \dots \cup S_n$. Since we have that $\mathfrak{S}_{\sigma}(F) = \mathfrak{S}_{\sigma}(G)$, it follows that $(S_1, \dots, S_n) \in \mathfrak{S}_{\sigma}(G)$ and thus $E \in \sigma(G)$, which contradicts our earlier assumption. Clearly, the same holds for any $E \in \sigma(G)$ with $E \notin \sigma(F)$.

Thus, we have shown that serialisation equivalence implies standard equivalence for every serialisable semantics σ. ☐

Proposition 2. *Let F, G be argumentation frameworks. It holds that $\mathsf{ad}(F) = \mathsf{ad}(G)$ if and only if $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G)$.*

Proof. We show both directions of the above statement as follows.

"⇐" This follows directly from Theorem 4.

"⇒" For this direction we show, w.l.o.g, that from $\mathcal{S} = (S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{ad}}(F)$ it follows that $\mathcal{S} \in \mathfrak{S}_{\mathsf{ad}}(G)$, inductively by the length of the sequence $\mathcal{S}$.

$(n = 1)$ For a serialisation sequence $(S_1) \in \mathfrak{S}_{\mathsf{ad}}(F)$, we know that $S_1 \in \mathsf{IS}(F)$ and thus $S_1 \in \mathsf{ad}(F)$. Because of $\mathsf{ad}(F) =$

$\mathsf{ad}(G)$, it follows that $S_1 \in \mathsf{ad}(G)$. Clearly, we must have $S_1 \in \mathsf{IS}(G)$ because there cannot exist any $S' \in \mathsf{ad}(G)$ with $S' \subsetneq S_1$ and $S' \notin \mathsf{ad}(F)$. Then, we must have that $(S_1) \in \mathfrak{S}_{\mathsf{ad}(G)}$.

$(n \Rightarrow n+1)$ Let $(S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{ad}}(F)$ be an admissible serialisation sequence of F. It holds that $(S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{ad}}(G)$. Consider the serialisation sequence $\mathcal{S} = (S_1, \dots, S_{n+1}) \in \mathfrak{S}_{\mathsf{ad}}(F)$. Assume that $\mathcal{S} \notin \mathfrak{S}_{\mathsf{ad}}(G)$. Because of $(S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{ad}}(G)$ it follows that we must have $S_{n+1} \notin \mathsf{IS}(G^{S_1 \cup \dots \cup S_n})$. However, since $\mathsf{ad}(F) = \mathsf{ad}(G)$ we know that $S_1 \cup \dots \cup S_n \in \mathsf{ad}(G)$ and also $S_1 \cup \dots \cup S_{n+1} \in \mathsf{ad}(G)$. That means there exists some admissible serialisation sequence $(S_1, \dots, S_n, T_1, \dots, T_m)$ in G with $T_1 \cup \dots \cup T_m = S_{n+1}$. It follows that $(S_1, \dots, S_n, T_1) \in \mathfrak{S}_{\mathsf{ad}}(G)$ and thus $S_1 \cup \dots \cup S_n \cup T_1 \in \mathsf{ad}(G)$. Via $\mathsf{ad}(F) = \mathsf{ad}(G)$ it follows that $S_1 \cup \dots \cup S_n \cup T_1 \in \mathsf{ad}(F)$. However, then we must have $T_1 \in \mathsf{IS}(F^{S_1 \cup \dots \cup S_n})$.This leads to a contradiction because $T_1 \subsetneq S_{n+1}$ and $S_{n+1} \in \mathsf{IS}(F^{S_1 \cup \dots \cup S_n})$. Hence, we have that $\mathcal{S} \in \mathfrak{S}_{\mathsf{ad}}(G)$.

To summarise, we have shown it follows from $\mathsf{ad}(F) = \mathsf{ad}(G)$ that $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G)$. Together with the other direction, that means it holds that $\mathsf{ad}(F) = \mathsf{ad}(G)$ if and only if $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(G)$. □

Proposition 3. *Let F, G be argumentation frameworks. It holds that $\mathsf{sa}(F) = \mathsf{sa}(G)$ if and only if $\mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(G)$.*

Proof. Analogously to the proof of Proposition 2, we show both directions as follows.

"⇐" This follows directly from Theorem 4.

"⇒" For this direction we show, w.l.o.g, that from $\mathcal{S} = (S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{sa}}(F)$ it follows that $\mathcal{S} \in \mathfrak{S}_{\mathsf{sa}}(G)$, inductively by the length of the sequence $\mathcal{S}$.

$(n = 1)$ For a serialisation sequence $(S_1) \in \mathfrak{S}_{\mathsf{sa}}(F)$, we know that $S_1 \in \mathsf{IS}^{\not\leftarrow}(F)$ and thus $S_1 \in \mathsf{sa}(F)$. Because of $\mathsf{sa}(F) = \mathsf{sa}(G)$, it follows that $S_1 \in \mathsf{sa}(G)$. Clearly, we must have $S_1 \in \mathsf{IS}^{\not\leftarrow}(G)$ because there cannot exist any $S' \in \mathsf{sa}(G)$ with $S' \subsetneq S_1$ and $S' \notin \mathsf{sa}(F)$. Then, we must have that $(S_1) \in \mathfrak{S}_{\mathsf{sa}(G)}$.

$(n \Rightarrow n+1)$ Let $(S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{sa}}(F)$ be a strongly admissible serialisation sequence of F. It holds that $(S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{sa}}(G)$. Consider the serialisation sequence $\mathcal{S} = (S_1, \dots, S_{n+1}) \in \mathfrak{S}_{\mathsf{sa}}(F)$. Because of $\mathsf{sa}(F) = \mathsf{sa}(G)$ we know that $S_1 \cup \dots \cup S_n \in \mathsf{sa}(G)$ and also $S_1 \cup \dots \cup S_{n+1} \in \mathsf{sa}(G)$. Recall that for all unattacked initial sets $S \in \mathsf{IS}^{\not\leftarrow}(H)$ for any AF H it holds that $|S| = 1$. Because of $(S_1, \dots S_n) \in \mathfrak{S}_{\mathsf{sa}}(G)$ it now follows directly that we must have $S_{n+1} \in \mathsf{IS}^{\not\leftarrow}(G^{S_1 \cup \dots \cup S_n})$, since there can exist no $S' \in \mathsf{IS}^{\not\leftarrow}(G^{S_1 \cup \dots \cup S_n})$ with $S' \subseteq S_{n+1}$. Hence, it follows that $(S_1, \dots, S_{n+1}) \in \mathfrak{S}_{\mathsf{ad}}(G)$.

To summarise, we have shown it follows from $\mathsf{sa}(F) = \mathsf{sa}(G)$ that $\mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(G)$. Together with the other direction, that means it holds that $\mathsf{sa}(F) = \mathsf{sa}(G)$ if and only if $\mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(G)$. □

Proposition 4. *Let F, G be argumentation frameworks. It holds that*

$$F \equiv^s_{ad} G \text{ iff } F \equiv^s_{pr} G \text{ iff } F \equiv^s_{uc} G.$$

Proof. We only show the statement: $F \equiv^s_{\mathsf{ad}} G$ iff $F \equiv^s_{\mathsf{uc}} G$. The rest follows from Theorem 2. We show both directions of the statement $F \equiv^s_{\mathsf{ad}} G$ iff $F \equiv^s_{\mathsf{uc}} G$ individually as follows.

"⇒" Let F, G be argumentation frameworks. From $F \equiv^s_{\mathsf{ad}} G$, it follows per Definition 7 that $\mathsf{ad}(F \cup H) = \mathsf{ad}(G \cup H)$, for all AFs H. In other words, we have that $F \cup H \equiv_{\mathsf{ad}} G \cup H$ for all AFs H. Per Theorem 3, it follows directly that $F \cup H \equiv_{\mathsf{uc}} G \cup H$ and thus $\mathsf{uc}(F \cup H) = \mathsf{uc}(G \cup H)$. Clearly, it follows $F \equiv^s_{\mathsf{uc}} G$.

"⇐" Let $F = (\mathcal{A}_F, \mathcal{R}_F), G = (\mathcal{A}_G, \mathcal{R}_G)$ be argumentation frameworks. From $F \equiv^s_{\mathsf{uc}} G$ it follows that $\mathsf{uc}(F \cup H) = \mathsf{uc}(G \cup H)$, for every AF H. Let $H = (\mathcal{A}_H, \mathcal{R}_H)$ be some arbitrary AF. We show that for every $S \in \mathsf{ad}(F \cup H)$ it follows that $S \in \mathsf{ad}(G \cup H)$ (the other direction follows analogously). So for some $S \in \mathsf{ad}(F \cup H)$ we have to show the following two conditions hold for S in $G \cup H$:

(1) S is conflict-free in $G \cup H$, and

(2) S defends all its members in $G \cup H$.

In the following, we will show that for every case: if the statement $S \in \mathsf{ad}(G \cup H)$ does not hold then there exists a construction for some AF H' such that $\mathsf{uc}(F \cup H') \neq \mathsf{uc}(G \cup H')$ which contradicts the initial assumption $F \equiv^s_{\mathsf{uc}} G$.

to (1): Assume the contrary, i. e., S is not conflict-free in $G \cup H$. That means there exist arguments $a, b \in S$ with $(a, b) \in \mathcal{R}_{G \cup H}$. We distinguish between two cases:

(1.1) $a = b$,

(1.2) $a \neq b$.

to (1.1): We have that $(a, a) \in \mathcal{R}_{G \cup H}$ and $(a, a) \notin \mathcal{R}_{F \cup H}$. It follows directly that $(a, a) \in \mathcal{R}_G$ and $(a, a) \notin \mathcal{R}_F$. Consider the AF $H' = (\mathcal{A}_{H'}, \mathcal{R}_{H'})$ with

$$\mathcal{A}_{H'} = \mathcal{A}_F \cup \mathcal{A}_G$$
$$\mathcal{R}_{H'} = \{(a, a'), (a', a') \mid a' \in \mathcal{A}_{H'} \setminus \{a\}\}$$

In words, in the AF H' every argument of F and G, except a, attacks itself and is also attacked by a. Clearly, we have $\mathsf{uc}(F \cup H') = \{a\}$ and $\mathsf{uc}(G \cup H') = \{\emptyset\}$, which contradicts the assumption that F and G are strongly equivalent wrt. unchallenged semantics.

to (1.2): We have $(a, b) \notin \mathcal{R}_{F \cup H}$ since $S \in \mathsf{ad}(F \cup H)$. Furthermore, it follows directly that $(a, b) \notin \mathcal{R}_F$ and $(a, b) \in \mathcal{R}_G$. Consider the AF $H' = (\mathcal{A}_{H'}, \mathcal{R}_{H'})$ with

$$\mathcal{A}_{H'} = \mathcal{A}_F \cup \mathcal{A}_G$$
$$\mathcal{R}_{H'} = \{(a', a'), (a', a) \mid a' \in \mathcal{A}_{H'} \setminus \{a\}\} \cup \{(a, a') \mid a' \in \mathcal{A}_{H'} \setminus \{a, b\}\}$$

In words, in the AF H' every argument of F and G, except a, attacks itself and a, while a attacks every other argument except b (and itself). Now we have that $\mathsf{uc}(F \cup H') = \{\emptyset\}$ and because of $(a, b) \in \mathcal{R}_G$ we have $\mathsf{uc}(G \cup H') = \{\{a\}\}$, which contradicts the assumption that F and G are strongly equivalent wrt. unchallenged semantics. Hence, S must be conflict-free in $G \cup H$.

to (2): Assume the contrary, i. e., there is some $a \in \mathcal{A}_{G \cup H}$ with $a\mathcal{R}_{G \cup H} S$ and $S \not\mathcal{R}_{G \cup H} a$. Clearly, if $a \in S$ it follows from the construction for (1) that we have a contradiction. So, we have that $a \notin S$. Let $b \in S$ be the undefended argument. We have that $(a, b) \in \mathcal{R}_{G \cup H}$ and $(c, a) \notin \mathcal{R}_{G \cup H}$ for all $c \in S$.

Now, one of two cases must apply:

(2.1) $(a, b) \in \mathcal{R}_{F \cup H}$, or

(2.2) $(a, b) \notin \mathcal{R}_{F \cup H}$.

to (2.1): Because of $S \in \mathsf{ad}(F \cup H)$, there exists some $c \in S$ with $(c, a) \in \mathcal{R}_{F \cup H}$. Recall that $(c, a) \notin \mathcal{R}_{G \cup H}$. It follows directly that $(c, a) \notin \mathcal{R}_G$ and $(c, a) \in \mathcal{R}_F$. Clearly, a construction of H' analogous to that in (1.2) leads to a contradiction.
to (2.2): We have that $(a, b) \notin \mathcal{R}_{F \cup H}$. We have $(b, b) \notin \mathcal{R}_F$ since $b \in S$ and $S \in \mathsf{ad}(F \cup H)$. We further distinguish between two cases:

(2.2.1) $(a, a) \in \mathcal{R}_{G \cup H}$,

(2.2.2) $(a, a) \notin \mathcal{R}_{G \cup H}$.

to (2.2.1): We have that $(a, b) \in \mathcal{R}_G$ and $(a, b) \notin \mathcal{R}_F$. Clearly, if $(a, a) \notin \mathcal{R}_F$, we can construct some H' like in (1.1) to arrive at a contradiction. So we assume $(a, a) \in \mathcal{R}_F$. Consider the AF $H' = (\mathcal{A}_{H'}, \mathcal{R}_{H'})$ with

$$\mathcal{A}_{H'} = \mathcal{A}_F \cup \mathcal{A}_G$$
$$\mathcal{R}_{H'} = \{(a', a'), (b, a') \mid a' \in \mathcal{A}_{H'} \setminus \{a, b\}\}$$

In words, in H' all arguments, except a and b attack themselves and are attacked by b. We have that $\mathsf{uc}(F \cup H') = \{\{b\}\}$ and $\mathsf{uc}(G \cup H') = \{\emptyset\}$ since b does not defend itself against a in $G \cup H'$.
to (2.2.2): Here, a construction of H' analogous to that of (1.2) leads to a contradiction since $(a, b) \in \mathcal{R}_G$ and $(a, b) \notin \mathcal{R}_F$.
It follows that S must defend all its members in $G \cup H$ and thus S is admissible in $G \cup H$.
Therefore, we have that from $\mathsf{uc}(F \cup H) = \mathsf{uc}(G \cup H)$ it follows necessarily that $\mathsf{ad}(F \cup H) = \mathsf{ad}(G \cup H)$. □

Lemma 1. *Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework. Then the following statements hold*

1. $\mathfrak{S}_{ad}(F) = \mathfrak{S}_{ad}(F^{ak})$,
2. $\mathfrak{S}_{pr}(F) = \mathfrak{S}_{pr}(F^{ak})$,
3. $\mathfrak{S}_{uc}(F) = \mathfrak{S}_{uc}(F^{ak})$,
4. $\mathfrak{S}_{sa}(F) = \mathfrak{S}_{sa}(F^{gk})$,
5. $\mathfrak{S}_{gr}(F) = \mathfrak{S}_{gr}(F^{gk})$,
6. $\mathfrak{S}_{co}(F) = \mathfrak{S}_{co}(F^{co})$.

Proof. We show each of the above statements individually as follows:

- $\mathfrak{S}_{ad}(F) = \mathfrak{S}_{ad}(F^{ak})$.
  Per Corollary 1, we know that $\mathsf{ad}(F) = \mathsf{ad}(F^{ak})$ holds. Thus, together with Proposition 2 it follows directly that $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(F^{ak})$.
- $\mathfrak{S}_{pr}(F) = \mathfrak{S}_{pr}(F^{ak})$.
  We operate under the ad-kernel, thus from Corollary 1 it follows that $\mathsf{ad}(F) = \mathsf{ad}(F^{ak})$ and $\mathsf{pr}(F) = \mathsf{pr}(F^{ak})$. From the above proof, we know that $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(F^{ak})$. Recall that every preferred serialisation sequence $(S_1, \ldots, S_n)$ of F is also by definition an admissible serialisation sequence of F. Thus, $(S_1, \ldots, S_n)$ is an admissible serialisation sequence in the kernel framework F^{ak}. Assume $(S_1, \ldots, S_n)$ were not a preferred serialisation sequence in the kernel framework F^{ak}. That would mean there exists a superset $E' \supset S_1 \cup \cdots \cup S_n$ which is a preferred extension of the kernel framework F^{ak}, but not of F. That is not possible, since we have $\mathsf{pr}(F) = \mathsf{pr}(F^{ak})$ and thus the lemma holds for the preferred semantics.
- $\mathfrak{S}_{uc}(F) = \mathfrak{S}_{uc}(F^{ak})$.
  Per Corollary 1 we have that $\mathsf{ad}(F) = \mathsf{ad}(F^{ak})$. From Proposition 2, it follows directly that $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(F^{ak})$, i. e., we have $F \equiv^{se}_{\mathsf{ad}} F^{ak}$. It follows further from Theorem 3 that we have $F \equiv^{se}_{\mathsf{uc}} F^{ak}$ and thus $\mathfrak{S}_{\mathsf{uc}}(F) = \mathfrak{S}_{\mathsf{uc}}(F^{ak})$.
- $\mathfrak{S}_{sa}(F) = \mathfrak{S}_{sa}(F^{gk})$.
  Per Corollary 1, we know that $\mathsf{sa}(F) = \mathsf{sa}(F^{ak})$ holds. Thus, together with Proposition 3 it follows directly that $\mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(F^{gk})$.
- $\mathfrak{S}_{gr}(F) = \mathfrak{S}_{gr}(F^{gk})$.
  Follows directly from the above proofs. From $\mathfrak{S}_{\mathsf{sa}}(F) = \mathfrak{S}_{\mathsf{sa}}(F^{gk})$ we can show analogously to the proof for the preferred semantics that the statement holds. In particular, from the fact that the set of grounded serialisation sequences is exactly the set of those strongly admissible serialisation sequences corresponding to the subset-maximal strongly admissible extension (which is the grounded extension).
- $\mathfrak{S}_{co}(F) = \mathfrak{S}_{co}(F^{co})$.
  Consider the *co*-kernel from Equation (4). We now show that any attack that is removed in the kernel framework F^{ck} does not influence the set of complete serialisation sequences $\mathfrak{S}_{\mathsf{co}}(F^{ck})$. Assume there exist arguments $a, b \in \mathcal{A}$ with $(a, b) \in \mathcal{R}$ such that $a \neq b$, $(a, a) \in \mathcal{R}$ and $(b, b) \in \mathcal{R}$. For the kernel framework F^{ck} we then have that $(a, a) \in \mathcal{R}^{ck}$, $(b, b) \in \mathcal{R}^{ck}$ while $(a, b) \notin \mathcal{R}^{ck}$. Since a and b are self-attacking, it follows directly that for all complete extensions $E \in \mathsf{co}(F)$ we have that $a \notin E$ and $b \notin E$. The same holds for all $E \in \mathsf{co}(F^{ck})$. Then, there can exist no S_i with $a \in S_i$ or $b \in S_i$ with $1 \leq i \leq n$ such that $(S_1, \ldots S_n)$ is a complete serialisation sequence. Furthermore, this means while a attacks b in F it can never defend some argument $c \in \mathcal{A}$ in F. Thus, the removal of (a, b) in F^{ck} does not remove any possible defense between a and c. It follows, that the lemma holds for the complete semantics. □

Theorem 5. *Let $\sigma \in \{ad, pr, sa, gr, co\}$ be a semantics. For any two argumentation frameworks F and G, if $F \equiv^{s}_{\sigma} G$, then it follows that $F \equiv^{se}_{\sigma} G$.*

Proof. Let $\sigma \in \{\mathsf{ad}, \mathsf{pr}, \mathsf{sa}, \mathsf{gr}, \mathsf{co}\}$ be a semantics. We show that if two argumentation frameworks F and G are strongly equivalent wrt. σ, then they are also serialisation equivalent wrt. σ, i. e., it follows from $F \equiv^{s}_{\sigma} G$, that $F \equiv^{se}_{\sigma} G$.

Assume that $F \equiv^{s}_{\sigma} G$. Then it follows from Theorem 1 that $F^k = G^k$, for any kernel $k \in \{sk, ak, gk, ck\}$. That leaves us to show that $\mathfrak{S}_\sigma(F) = \mathfrak{S}_\sigma(F^k)$, with k being the respective kernel for the semantics σ, for any argumentation framework F. This follows for any semantics $\sigma \in \{\mathsf{ad}, \mathsf{pr}, \mathsf{sa}, \mathsf{gr}, \mathsf{co}\}$ directly from Lemma 1. □

Proposition 5. *Let F an G be argumentation frameworks. Then $F \equiv^{se}_{st} G$ does not generally imply $F \equiv^{s}_{st}$ and vice versa.*

Proof. Follows directly from Example 14 and Example 13. □

Theorem 6.

1. *Eq^{se}_{σ} is coNP-complete, for $\sigma \in \{ad, pr, co, st\}$.*
2. *Eq^{se}_{σ} is in P, for $\sigma \in \{sa, gr\}$.*
3. *Eq^{se}_{uc} is Π^P_2-complete.*

Proof. We proof each statement individually as follows:

- *Eq^{se}_{ad} is coNP-complete.*
  For coNP-membership, consider the following non-deterministic algorithm for verifying that two given argumentation frameworks $F_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $F_2 = (\mathcal{A}_2, \mathcal{R}_2)$ are *not* serialisation equivalent. We first guess a sequence $\mathcal{S} = (S_1, \ldots S_n)$ with $S_1, \ldots, S_n \subseteq \mathcal{A}_1$ and verify for each S_i ($i = 1, \ldots, n$) that S_i is an initial set of $F_1^{S_1 \cup \ldots \cup S_{i-1}}$, which can be done in polynomial

time [4]. We then verify for each S_i ($i = 1, \ldots, n$) that S_i is an initial set of $F_2^{S_1 \cup \ldots \cup S_{i-1}}$. If one of the latter verification steps fails, $(S_1, \ldots, S_n)$ is then shown to be a serialisation sequence of F_1 but not F_2 and it follows $F_1 \not\equiv^{se}_{\mathsf{ad}} F_2$.

For coNP-hardness, we use the complementary problem of deciding credulous acceptance wrt. admissibility *coCRED*$_{\mathsf{ad}}$. Since the problem of deciding whether there is an admissible set S with $a \in S$ is NP-complete, see e. g. [2], the problem *coCRED*$_{\mathsf{ad}}$ is naturally coNP-complete. Let $F = (\mathcal{A}, \mathcal{R})$ and $a \in \mathcal{A}$ be an instance of *coCRED*$_{\mathsf{ad}}$. Let b be an argument with $b \notin \mathcal{A}$ and define $F' = (\mathcal{A}', \mathcal{R}')$ via

$$\begin{aligned}\mathcal{A}' &= \mathcal{A} \setminus \{a\} \cup \{b\} \\ \mathcal{R}' &= \mathcal{R} \setminus \{(a, c), (c, a) \mid c \in \mathcal{A}\} \cup \{(b, c) \mid (a, c) \in \mathcal{R}\} \cup \\ &\quad \{(c, b) \mid (c, a) \in \mathcal{R}\}\end{aligned}$$

In other words, F' is the same as F where a has been renamed to b. We claim that a is not contained in an admissible set of F iff $F \equiv^{se}_{\mathsf{ad}} F'$. We show both directions of the above statement individually.

"⇒" We know that a is not contained in any admissible set of F. Assume this direction does not hold, i. e., we have that $F \not\equiv^{se}_{\mathsf{ad}} F'$. Then, w.l.o.g, there must be a serialisation sequence $\mathcal{S} = (S_1, \ldots S_n)$ with $\mathcal{S} \in \mathfrak{S}_{\mathsf{ad}}(F)$ and $\mathcal{S} \notin \mathfrak{S}_{\mathsf{ad}}(F')$, or vice versa. Recall that F' only differs from F in the renamed argument b. Clearly, that means all serialisation sequences that do not contain this argument will be the same for both frameworks. However, per our initial assumption, we know that a is not in any admissible set of F and thus b is not in any admissible set of F'. Hence, there can be no admissible serialisation sequence containing the argument, therefore both frameworks must have the same set of admissible serialisation sequences and are serialisation equivalent wrt. the admissible semantics.

"⇐" We know that $F \equiv^{se}_{\mathsf{ad}} F'$ and thus $\mathfrak{S}_{\mathsf{ad}}(F) = \mathfrak{S}_{\mathsf{ad}}(F')$. Assume this direction does not hold and we have that there exists some $E \in \mathsf{ad}(F)$ with $a \in E$. Then, there must exist at least one admissible serialisation sequence $(S_1, \ldots S_n)$ for F with $S_1 \cup \cdots \cup S_n = E$ and $a \in S_i$ for some $1 \leq i \leq n$. However, since a is renamed to b in F', it follows that $(S_1, \ldots S_n) \notin \mathfrak{S}_{\mathsf{ad}}(F')$ and therefore $F \not\equiv^{se}_{\mathsf{ad}} F'$, which is a contradiction.

- *Eq*$^{se}_{\mathsf{pr}}$ *is coNP-complete.*
  Follows directly from Theorem 3 and the above statement for the admissible semantics.
- *Eq*$^{se}_{\mathsf{gr}}$ *is in P.*
  Consider the following deterministic algorithm for verifying that two given argumentation frameworks $F_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $F_2 = (\mathcal{A}_2, \mathcal{R}_2)$ are serialisation equivalent wrt. grounded semantics. We consider the characteristic function $\Gamma_F : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ of an argumentation framework $F = (\mathcal{A}, \mathcal{R})$ defined as $\Gamma_F(S) = \{a \in \mathcal{A} \mid S \text{ defends } a\}$. Computing $\Gamma_F(S)$ for some arbitrary S can be done in polynomial time. Recall, that unattacked initial sets are always singleton sets. Per definition, $\Gamma_F(\emptyset)$ returns then exactly all arguments that are unattacked in F and thus correspond to the unattacked initial sets in F. We now verify in polynomial time that $\mathfrak{S}_{\mathsf{gr}}(F_1) = \mathfrak{S}_{\mathsf{gr}}(F_2)$ as follows: First, we verify that $\Gamma_{F_1}(\emptyset) = \Gamma_{F_2}(\emptyset)$. Then, for each argument $a \in \Gamma_{F_1}(\emptyset)$, we set $S_1 = \{a\}$ and verify that $\Gamma_{F_1}(S_1) = \Gamma_{F_2}(S_1)$. Analogously, we continue, i. e., for all arguments $a \in \Gamma_{F_1}(S_i)$ we set $S_{i+1} = \{a\}$ and verify that $\Gamma_{F_1}(S_1 \cup \cdots \cup S_{i+1}) = \Gamma_{F_2}(S_1 \cup \cdots \cup S_{i+1})$ until we reach the fixpoint of Γ_F. If the verification step fails, i. e., for some i there exists w.l.o.g. some $a \in \Gamma_{F_1}(S_1 \cup \cdots \cup S_i)$ with $a \notin \Gamma_{F_2}(S_1 \cup \cdots \cup S_i)$, then we have that $(S_1, \ldots, S_i, \{a\}) \in \mathfrak{S}_{\mathsf{gr}}(F_1)$ and $(S_1, \ldots, S_i, \{a\}) \notin \mathfrak{S}_{\mathsf{gr}}(F_2)$. Thus, it follows $\mathfrak{S}_{\mathsf{gr}}(F_1) \neq \mathfrak{S}_{\mathsf{gr}}(F_2)$ and we have that $F_1 \not\equiv^{se}_{\mathsf{gr}} F_2$.
- *Eq*$^{se}_{\mathsf{sa}}$ *is in P.*
  Follows directly from Theorem 3 and the above statement for the grounded semantics.
- *Eq*$^{se}_{\mathsf{co}}$ *is coNP-complete.*
  For coNP-membership, consider the following non-deterministic algorithm for verifying that two given argumentation frameworks $F_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $F_2 = (\mathcal{A}_2, \mathcal{R}_2)$ are *not* serialisation equivalent wrt. complete semantics. We first guess a sequence $\mathcal{S} = (S_1, \ldots S_n)$ with $S_1, \ldots, S_n \subseteq \mathcal{A}_1$ and verify in polynomial time, as described in the proof for the admissible semantics, that $\mathcal{S} \in \mathfrak{S}_{\mathsf{ad}}(F_1)$. Additionally, we verify in polynomial time that $\mathsf{IS}^{\not\leftarrow}(F_1^{S_1 \cup \cdots \cup S_n}) = \emptyset$ via the characteristic function, i. e., we verify that $\Gamma_{F_1}(S_1 \cup \cdots \cup S_n) = S_1 \cup \cdots \cup S_n$. We then verify for each S_i ($i = 1, \ldots, n$) that S_i is an initial set of $F_2^{S_1 \cup \ldots \cup S_{i-1}}$ and that $\mathsf{IS}^{\not\leftarrow}(F_2^{S_1 \cup \cdots \cup S_n}) = \emptyset$. If one of the latter verification steps fails, $(S_1, \ldots, S_n)$ is then shown to be a complete serialisation sequence of F_1 but not F_2 and it follows $F_1 \not\equiv^{se}_{\mathsf{co}} F_2$.
  For coNP-hardness, we use the complementary problem of deciding credulous acceptance wrt. complete semantics *coCRED*$_{\mathsf{co}}$ and the proof is analogous to the hardness proof of the *Eq*$^{se}_{\mathsf{ad}}$ problem.
- *Eq*$^{se}_{\mathsf{st}}$ *is coNP-complete.*
  For coNP-membership, consider the following non-deterministic algorithm for verifying that two given argumentation frameworks $F_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $F_2 = (\mathcal{A}_2, \mathcal{R}_2)$ are *not* serialisation equivalent wrt. stable semantics. We first guess a sequence $\mathcal{S} = (S_1, \ldots S_n)$ with $S_1, \ldots, S_n \subseteq \mathcal{A}_1$ and verify in polynomial time, as described in the proof for the admissible semantics, that $\mathcal{S} \in \mathfrak{S}_{\mathsf{ad}}(F_1)$. Additionally, we verify in polynomial time that $F_1^{S_1 \cup \cdots \cup S_n} = (\emptyset, \emptyset)$. We then verify for each S_i ($i = 1, \ldots, n$) that S_i is an initial set of $F_2^{S_1 \cup \ldots \cup S_{i-1}}$ and that $F_2^{S_1 \cup \cdots \cup S_n} = (\emptyset, \emptyset)$. If one of the latter verification steps fails, $(S_1, \ldots, S_n)$ is then shown to be a stable serialisation sequence of F_1 but not F_2 and it follows $F_1 \not\equiv^{se}_{\mathsf{st}} F_2$.
  For coNP-hardness, we use the complementary problem of deciding credulous acceptance wrt. stable semantics *coCRED*$_{\mathsf{st}}$ and the proof is analogous to the hardness proof of the *Eq*$^{se}_{\mathsf{ad}}$ problem.
- *Eq*$^{se}_{\mathsf{uc}}$ *is* Π_2^P*-complete.*
  For Π_2^P-membership, we consider that the complement problem co*Eq*$^{se}_{\mathsf{uc}}$, i. e., the problem of deciding whether two given argumentation frameworks $F_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $F_2 = (\mathcal{A}_2, \mathcal{R}_2)$ are *not* serialisation equivalent wrt. unchallenged semantics. We show that $\neg$*Eq*$^{se}_{\mathsf{uc}}$ is in $\mathsf{NP}^{\mathsf{NP}} = \Sigma_2^P = \mathsf{co}\Pi_2^P$. For that we adapt the algorithm for deciding skeptical acceptance wrt. unchallenged semantics from [1]. More specifically, consider the following non-deterministic algorithm. We start by guessing an integer k. For $i = 1, \ldots, k$, we iteratively guess a set $S_i \subseteq \mathcal{A}$ and verify that $S_i \in \mathsf{IS}^{\not\leftarrow}(F_1^{S_1 \cup \ldots \cup S_{i-1}}) \cup \mathsf{IS}^{\not\leftrightarrow}(F_1^{S_1 \cup \ldots \cup S_{i-1}})$. The latter can be accomplished by an NP-oracle call, since the problem is coNP-complete [4]. Finally, we verify that $\mathsf{IS}^{\not\leftarrow}(F_1^{S_1 \cup \ldots \cup S_k}) = \mathsf{IS}^{\not\leftarrow}(F_1^{S_1 \cup \ldots \cup S_k}) = \emptyset$, which can be done in $\mathsf{P}^{\mathsf{NP}}_{\|} \subseteq \mathsf{NP}^{\mathsf{NP}}$ [4]. It follows that $(S_1, \ldots, S_k)$ is an unchallenged serialisation sequence of F_1. We then verify for each S_i ($i = 1, \ldots, k$) that S_i is an unattacked or unchallenged initial set of $F_2^{S_1 \cup \ldots \cup S_{i-1}}$ and that $\mathsf{IS}^{\not\leftarrow}(F_2^{S_1 \cup \ldots \cup S_k}) = \mathsf{IS}^{\not\leftarrow}(F_2^{S_1 \cup \ldots \cup S_k}) = \emptyset$. If one of the latter verification steps fails, $(S_1, \ldots, S_k)$ is then shown to be an un-

challenged serialisation sequence of F_1 but not F_2 and it follows $F_1 \not\equiv^{se}_{\mathsf{uc}} F_2$. This algorithm runs in Σ_2^P, so Eq^{se}_{uc} is in Π_2^P.
For Π_2^P-hardness, we use the complementary problem of deciding credulous acceptance wrt. unchallenged semantics *coCRED*$_{\mathsf{uc}}$ and the proof is analogous to the hardness proof of the Eq^{se}_{ad} problem.

□

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