On the Evaluation of Inconsistency Measures

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Abstract

We discuss the issue of evaluating inconsistency measures along the three dimensions of rationality postulates, expressivity, and computational complexity. We survey a broad selection on inconsistency measures and evaluate their performance on these three dimensions.

1 Introduction

In classical logic, the notion of *inconsistency* is a binary concept. Either a formula (or a set of formulas) is inconsistent or not. However, quantifying inconsistency is an important challenge for logical accounts to knowledge representation as differences in the severity of inconsistency may indeed be recognised for certain types of applications. Consider the following two classical logic knowledge bases

$$\mathcal{K}_1 = \{a, \neg a, b, \neg b\} \qquad\qquad \mathcal{K}_2 = \{\neg b, a, a \rightarrow b, c\}$$

Both $\mathcal{K}_1$ and $\mathcal{K}_2$ are classically inconsistent, i. e., there is no interpretation satisfying any of them. But looking closer into the structure of the knowledge bases one can identify differences in the severity of the inconsistency. In $\mathcal{K}_1$ there are two apparent contradictions, i. e., $\{a, \neg a\}$ and $\{b, \neg b\}$ are directly conflicting formulas. In $\mathcal{K}_2$, the conflict is a bit more hidden. Here, three formulas are necessary to produce a contradiction ($\{\neg b, a, a \rightarrow b\}$). Moreover, there is one formula in $\mathcal{K}_2$ (c), which is not participating in any conflict and one could still infer meaningful information from this by relying on e. g. paraconsistent reasoning techniques [5]. In conclusion, one should regard $\mathcal{K}_1$ as *more inconsistent* than $\mathcal{K}_2$.

The analysis of the severity of inconsistency in the knowledge bases $\mathcal{K}_1$ and $\mathcal{K}_2$ above was informal. Formal accounts to the problem of assessing the severity of inconsistency are given by *inconsistency measures* and there have been a lot of proposals of those in recent years (see the other chapters in this volume and the measures reviewed in Section 3). Up to today, the concept of *severity of inconsistency* has not been axiomatized in a satisfactory manner and the series of different inconsistency measures approach this challenge from different points of view and focus on different aspects on what constitutes *severity*. Consider the next two knowledge bases

$$\mathcal{K}_3 = \{a, \neg a, b\} \qquad \mathcal{K}_4 = \{a \vee b, \neg a \vee b, a \vee \neg b, \neg a \vee \neg b\}$$

Again both $\mathcal{K}_3$ and $\mathcal{K}_4$ are inconsistent, but which one is more inconsistent than the other? Our reasoning from above cannot be applied here in the same fashion. The knowledge base $\mathcal{K}_3$ contains an apparent contradiction ($\{a, \neg a\}$) but also a formula not participating in the inconsistency ($\{b\}$). The knowledge base $\mathcal{K}_4$ contains a "hidden" conflict as four formulas are necessary to produce a contradiction, but all formulas of $\mathcal{K}_4$ are participating in this. In this case, it is not clear how to quantitatively assess the inconsistency of these knowledge bases and different measures may order these knowledge bases differently. More generally speaking, it is not universally agreed upon which so-called *rationality postulates* should be satisfied by a reasonable account of inconsistency measurement, see [3] for a discussion. Besides concrete approaches to inconsistency measurement the community has also proposed a series of those rationality postulates in order to describe general desirable behaviour and the classification of inconsistency measures by the postulates they satisfy is still one the most important ways to evaluate the quality of a measure, even if the set of desirable postulates is not universally accepted. For example, one of the most popular rationality postulates is *monotonicity* which states that for any $\mathcal{K} \subseteq \mathcal{K}'$, the knowledge base $\mathcal{K}$ cannot be regarded as more inconsistent as $\mathcal{K}'$. The justification for this demand is that inconsistency cannot be resolved when adding new information but only increased. While this is usually regarded as a reasonable demand there are also situations where *monotonicity* may be seen as counterintuitive. Consider the next two knowledge bases

$$\mathcal{K}_5 = \{a, \neg a\} \qquad \mathcal{K}_6 = \{a, \neg a, b_1, \ldots, b_{998}\}$$

We have $\mathcal{K}_5 \subseteq \mathcal{K}_6$ and following *monotonicity*, $\mathcal{K}_6$ should be regarded as least as inconsistent as $\mathcal{K}_5$. However, when judging the content of the knowledge bases in a "relative" manner, $\mathcal{K}_5$ may seem more inconsistent. More precisely, $\mathcal{K}_5$ contains no useful information and all formulas of $\mathcal{K}_5$ are in conflict with another formula. In $\mathcal{K}_6$, however, only 2 out of 1000 formulas are participating in the contradiction. So it may also be a reasonable point of view to judge $\mathcal{K}_5$ more inconsistent than $\mathcal{K}_6$.

In this chapter, we will not give a final answer to the discussion on which rationality postulate is desirable or not. We will, however, provide a comprehensive overview of the existing rationality postulates and the compliance of different measures wrt. those, continuing work from [41]. It is up to the reader and future work to conclude said discussion. Besides satisfaction of rationality postulates we will address two more "objective" evaluation metrics, namely *expressivity* [38] and *computational complexity* [44, 43]. The former refers to the ability of an inconsistency measure to differentiate many levels of the severity of inconsistency. Consider the following family of knowledge bases

$$\mathcal{K}_7^1 = \{a_1, \neg a_1\} \quad \mathcal{K}_7^2 = \{a_1, \neg a_1, a_2, \neg a_2\} \quad \ldots \; \mathcal{K}_7^n = \{a_1, \neg a_1, \ldots, a_n, \neg a_n\}$$

Each knowledge base $\mathcal{K}_7^{i+1}$ contains one more apparent contradiction than $\mathcal{K}_7^i$, so it is reasonable to assess $\mathcal{K}_7^{i+1}$ as *strictly* more inconsistent than $\mathcal{K}_7^i$. Following [38] we will present a formal framework for assessing the expressivity of inconsistency measures and provide a comprehensive overview of the expressivity of different measures. Finally, we will consider the computational complexity involved in computing the value of an inconsistency measure, following [44, 43]. As detecting inconsistency alone is an intractable problem, we cannot hope to determine inconsistency values in an efficient manner. However, inconsistency measures can be classified into different levels of the polynomial hierarchy and thus algorithms determining them may exhibit significant differences in performance. As before, we provide a comprehensive overview of the computational complexity landscape of different measures as well.

This chapter summarises the works [38, 41, 44, 43] and complements their results by additionally considering more recent approaches to inconsistency measurement. The rest of this chapter is organised as follows. In Section 2 we present necessary preliminaries on propositional logic and we review our selection of inconsistency measures to be studied in Section 3. In Sections 4, 5, and 6 presents the evaluation measures of rationality postulates, expressivity, and computational complexity, respectively. We conclude in Section 7. Appendix 7 contains proofs of new technical results.

2 Preliminaries

Let At be some fixed propositional signature, i. e., a (possibly infinite) set of propositions, and let $\mathcal{L}(\mathsf{At})$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*conjunction*), $\vee$ (*disjunction*), $\rightarrow$ (*implication*), and $\neg$ (*negation*). A literal is a proposition p or a negated proposition $\neg p$.

Definition 1. A knowledge base $\mathcal{K}$ is a finite set of formulas $\mathcal{K} \subseteq \mathcal{L}(\mathsf{At})$. Let $\mathbb{K}$ be the set of all knowledge bases.

A clause is a disjunction of literals. A formula is in conjunctive normal form (CNF) if the formula is a conjunction of clauses. If X is a formula or a set of formulas we write $\mathsf{At}(X)$ to denote the set of propositions appearing in X. For a set $S = \{\phi_1, \ldots, \phi_n\}$ let $\bigwedge S = \phi_1 \wedge \ldots \wedge \phi_n$.

Semantics for a propositional language is given by *interpretations* where an *interpretation* ω on At is a function $\omega : \mathsf{At} \rightarrow \{\mathsf{true}, \mathsf{false}\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At. An interpretation ω *satisfies* (or is a *model* of) an atom $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \mathsf{true}$. The satisfaction relation $\models$ is extended to formulas in the usual way.

For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$. Define furthermore the set of models $\mathsf{Mod}(X) = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models X\}$ for every formula or set of formulas X. Two formulas or sets of formulas X_1, X_2 are *equivalent*, denoted by $X_1 \equiv X_2$, if and only if $\mathsf{Mod}(X_1) = \mathsf{Mod}(X_2)$. Furthermore, two sets of formulas X_1, X_2 are *semi-extensionally equivalent* [36, 37]—or *bijection equivalent* [10]—if and only if there is a bijection $s : X_1 \rightarrow X_2$ such that for all $\alpha \in X_1$ we have $\alpha \equiv s(\alpha)$. We denote this by $X_1 \equiv_b X_2$. If $\mathsf{Mod}(X) = \emptyset$ we also write $X \models \perp$ and say that X is *inconsistent*.

3 Inconsistency Measures

Let $\mathbb{R}^{\infty}_{\geq 0}$ be the set of non-negative real values including ∞. Inconsistency measures are functions $\mathcal{I} : \mathbb{K} \rightarrow \mathbb{R}^{\infty}_{\geq 0}$ that aim at assessing the severity of inconsistency in a knowledge base $\mathcal{K}$, cf. [10]. The basic idea is that the larger the inconsistency in $\mathcal{K}$ the larger the value $\mathcal{I}(\mathcal{K})$. Formally, we define inconsistency measures as follows, cf. e. g. [15].

Definition 2. An inconsistency measure $\mathcal{I}$ is any function $\mathcal{I} : \mathbb{K} \rightarrow \mathbb{R}^{\infty}_{\geq 0}$.

There is a wide variety of inconsistency measures in the literature. In this work, we select 22 inconsistency measures in order to discuss issues pertaining to their evaluation. We briefly introduce these measures in this section for the sake of completeness, but we refer for a detailed explanation to the corresponding original papers.[1]

The measure $\mathcal{I}_d(\mathcal{K})$ is usually referred to as a baseline for inconsistency measures as it only distinguishes between consistent and inconsistent knowledge bases.

Definition 3 ([15])**.** The *drastic inconsistency measure* $\mathcal{I}_d : \mathbb{K} \rightarrow \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_d(\mathcal{K}) = \begin{cases} 1 & \text{if } \mathcal{K} \models \perp \\ 0 & \text{otherwise} \end{cases}$$

for $\mathcal{K} \in \mathbb{K}$.

A set $M \subseteq \mathcal{K}$ is called a *minimal inconsistent subset* (MI) of $\mathcal{K}$ if $M \models \perp$ and there is no $M' \subset M$ with $M' \models \perp$. Let $\mathsf{MI}(\mathcal{K})$ be the set of all MIs of $\mathcal{K}$.

[1] Implementations of these measures can also be found in the *Tweety Libraries for Artificial Intelligence* [42] and an online interface is available at `http://tweetyproject.org/w/incmes`

Definition 4 ([15])**.** The MI-*inconsistency measure* $\mathcal{I}_{\mathsf{MI}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = |\mathsf{MI}(\mathcal{K})|$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 5 ([15])**.** The MI^c-*inconsistency measure* $\mathcal{I}_{\mathsf{MI^c}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{\mathsf{MI^c}}(\mathcal{K}) = \sum_{M \in \mathsf{MI}(\mathcal{K})} \frac{1}{|M|}$$

for $\mathcal{K} \in \mathbb{K}$.

A *probability function* P on $\mathcal{L}(\mathsf{At})$ is a function $P : \Omega(\mathsf{At}) \to [0,1]$ with $\sum_{\omega \in \Omega(\mathsf{At})} P(\omega) = 1$. We extend P to assign a probability to any formula $\phi \in \mathcal{L}(\mathsf{At})$ by defining

$$P(\phi) = \sum_{\omega \models \phi} P(\omega)$$

Let $\mathcal{P}(\mathsf{At})$ be the set of all those probability functions.

Definition 6 ([22])**.** The η-*inconsistency measure* $\mathcal{I}_{\eta} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{\eta}(\mathcal{K}) = 1 - \max\{\xi \mid \exists P \in \mathcal{P}(\mathsf{At}) : \forall \alpha \in \mathcal{K} : P(\alpha) \geq \xi\}$$

for $\mathcal{K} \in \mathbb{K}$.

A *three-valued interpretation* υ on At is a function $\upsilon : \mathsf{At} \to \{T, F, B\}$ where the values T and F correspond to the classical true and false, respectively. The additional truth value B stands for *both* and is meant to represent a conflicting truth value for a proposition. The function υ is extended to arbitrary formulas as shown in Table 1. An interpretation υ satisfies a formula α, denoted by $\upsilon \models^3 \alpha$ if either $\upsilon(\alpha) = T$ or $\upsilon(\alpha) = B$. Inconsistency can be measured by seeking an interpretation υ that assigns B to a minimal number of propositions.

Definition 7 ([10])**.** The *contension inconsistency measure* $\mathcal{I}_c : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_c(\mathcal{K}) = \min\{|\upsilon^{-1}(B)| \mid \upsilon \models^3 \mathcal{K}\}$$

for $\mathcal{K} \in \mathbb{K}$.

Let $\mathsf{MC}(\mathcal{K})$ be the set of maximal consistent subsets of $\mathcal{K}$, i.e.

$$\mathsf{MC}(\mathcal{K}) = \{\mathcal{K}' \subseteq \mathcal{K} \mid \mathcal{K}' \not\models \perp \wedge \forall \mathcal{K}'' \supsetneq \mathcal{K}' : \mathcal{K}'' \models \perp\}$$

Furthermore, let $\mathsf{SC}(\mathcal{K})$ be the set of self-contradictory formulas of $\mathcal{K}$, i.e.

$$\mathsf{SC}(\mathcal{K}) = \{\phi \in \mathcal{K} \mid \phi \models \perp\}$$

α	β	$\upsilon(\alpha \wedge \beta)$	$\upsilon(\alpha \vee \beta)$
T	T	T	T
T	B	B	T
T	F	F	T
B	T	B	T
B	B	B	B
B	F	F	B
F	T	F	T
F	B	F	B
F	F	F	F

α	$\upsilon(\neg\alpha)$
T	F
B	B
F	T

Table 1: Truth tables for propositional three-valued logic.

Definition 8 ([10])**.** The MC-*inconsistency measure* $\mathcal{I}_{mc} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{mc}(\mathcal{K}) = |\mathsf{MC}(\mathcal{K})| + |\mathsf{SC}(\mathcal{K})| - 1$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 9 ([10])**.** The *problematic inconsistency measure* $\mathcal{I}_p : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_p(\mathcal{K}) = |\bigcup_{M \in \mathsf{MI}(\mathcal{K})} M|$$

for $\mathcal{K} \in \mathbb{K}$.

A subset $H \subseteq \Omega(\mathsf{At})$ is called a *hitting set* of $\mathcal{K}$ if for every $\phi \in \mathcal{K}$ there is $\omega \in H$ with $\omega \models \phi$.

Definition 10 ([39])**.** The *hitting-set inconsistency measure* $\mathcal{I}_{hs} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{hs}(\mathcal{K}) = \min\{|H| \mid H \text{ is a hitting set of } \mathcal{K}\} - 1$$

for $\mathcal{K} \in \mathbb{K}$ with $\min \emptyset = \infty$.

An *interpretation distance* d is a function $d : \Omega(\mathsf{At}) \times \Omega(\mathsf{At}) \to [0, \infty)$ that satisfies (let $\omega, \omega', \omega'' \in \Omega(\mathsf{At})$)

1. $d(\omega, \omega') = 0$ if and only if $\omega = \omega'$ (*reflexivity*),
2. $d(\omega, \omega') = d(\omega', \omega)$ (*symmetry*), and
3. $d(\omega, \omega'') \leq d(\omega, \omega') + d(\omega', \omega'')$ (*triangle inequality*).

One prominent example of such a distance is the *Dalal distance* d_{d} defined via

$$d_{\mathrm{d}}(\omega,\omega') = |\{a \in \mathsf{At} \mid \omega(a) \neq \omega'(a)\}|$$

for all $\omega,\omega' \in \Omega(\mathsf{At})$. If $X \subseteq \Omega(\mathsf{At})$ is a set of interpretations we define $d_{\mathrm{d}}(X,\omega) = \min_{\omega' \in X} d_{\mathrm{d}}(\omega',\omega)$ (if $X = \emptyset$ we define $d_{\mathrm{d}}(X,\omega) = \infty$). For definitions 11, 12, and 13 below we assume d_{d} fixed but the measures could be defined using arbitrary distances.

Definition 11 ([12])**.** The Σ-*distance inconsistency measure* $\mathcal{I}^{\Sigma}_{\mathrm{dalal}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}^{\Sigma}_{\mathrm{dalal}}(\mathcal{K}) = \min\left\{\sum_{\alpha\in\mathcal{K}} d_{\mathrm{d}}(\mathsf{Mod}(\alpha),\omega) \mid \omega \in \Omega(\mathsf{At})\right\}$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 12 ([12])**.** The max-*distance inconsistency measure* $\mathcal{I}^{\max}_{\mathrm{dalal}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}^{\max}_{\mathrm{dalal}}(\mathcal{K}) = \min\left\{\max_{\alpha\in\mathcal{K}} d_{\mathrm{d}}(\mathsf{Mod}(\alpha),\omega) \mid \omega \in \Omega(\mathsf{At})\right\}$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 13 ([12])**.** The *hit-distance inconsistency measure* $\mathcal{I}^{\mathrm{hit}}_{\mathrm{dalal}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}^{\mathrm{hit}}_{\mathrm{dalal}}(\mathcal{K}) = \min\left\{|\{\alpha \in \mathcal{K} \mid d_{\mathrm{d}}(\mathsf{Mod}(\alpha),\omega) > 0\}| \mid \omega \in \Omega(\mathsf{At})\right\}$$

for $\mathcal{K} \in \mathbb{K}$.

For $\mathcal{K} \in \mathbb{K}$ define

$$\mathsf{MI}^{(i)}(\mathcal{K}) = \{M \in \mathsf{MI}(\mathcal{K}) \mid |M| = i\}$$

$$\mathsf{CN}^{(i)}(\mathcal{K}) = \{C \subseteq \mathcal{K} \mid |C| = i \wedge C \not\models \perp\}$$

$$R_i(\mathcal{K}) = \begin{cases} 0 & \text{if } |\mathsf{MI}^{(i)}(\mathcal{K})| + |\mathsf{CN}^{(i)}(\mathcal{K})| = 0 \\ |\mathsf{MI}^{(i)}(\mathcal{K})|/(|\mathsf{MI}^{(i)}(\mathcal{K})| + |\mathsf{CN}^{(i)}(\mathcal{K})|) & \text{otherwise} \end{cases}$$

for $i = 1,\ldots,|\mathcal{K}|$.

Definition 14 ([31])**.** The D_f*-inconsistency measure* $\mathcal{I}_{D_f} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{D_f}(\mathcal{K}) = 1 - \Pi_{i=1}^{|\mathcal{K}|}(1 - R_i(\mathcal{K})/i)$$

for $\mathcal{K} \in \mathbb{K}$.

A *minimal proof* for $\alpha \in \{x, \neg x \mid x \in \mathsf{At}\}$ in $\mathcal{K}$ is a set $\pi \subseteq \mathcal{K}$ such that

1. α appears as a literal in π
2. $\pi \models \alpha$, and
3. π is minimal wrt. set inclusion.

Let $P_m^{\mathcal{K}}(x)$ be the set of all minimal proofs of x in $\mathcal{K}$.

Definition 15 ([21])**.** The *proof-based inconsistency measure* $\mathcal{I}_{P_m} : \mathbb{K} \to \mathbb{R}_{\geq 0}^{\infty}$ is defined as

$$\mathcal{I}_{P_m}(\mathcal{K}) = \sum_{a \in \mathsf{At}} |P_m^{\mathcal{K}}(a)| \cdot |P_m^{\mathcal{K}}(\neg a)|$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 16 ([46])**.** The *mv inconsistency measure* $\mathcal{I}_{mv} : \mathbb{K} \to \mathbb{R}_{\geq 0}^{\infty}$ is defined as

$$\mathcal{I}_{mv}(\mathcal{K}) = \frac{|\bigcup_{M \in \mathsf{MI}(\mathcal{K})} \mathsf{At}(M)|}{|\mathsf{At}(\mathcal{K})|}$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 17 ([7])**.** The *nc-inconsistency measure* $\mathcal{I}_{nc} : \mathbb{K} \to \mathbb{R}_{\geq 0}^{\infty}$ is defined as

$$\mathcal{I}_{nc}(\mathcal{K}) = |\mathcal{K}| - \max\{n \mid \forall \mathcal{K}' \subseteq \mathcal{K} : |\mathcal{K}'| = n \Rightarrow \mathcal{K}' \not\models \perp\}$$

for $\mathcal{K} \in \mathbb{K}$.

The work [40] considers different families of inconsistency measures based on many-valued logics. We focus here on the three instantiations $\mathcal{I}_{t_{\text{prod}}}^{\text{fuz}}$, $\mathcal{I}_{t_{\min}}^{\text{fuz},\Sigma}$, $\mathcal{I}_{t_{\text{prod}}}^{\text{fuz},\Sigma}$ based on Fuzzy logic.

A *fuzzy product interpretation* ω is a function $\omega : \mathcal{L}(\mathsf{At}) \to [0, 1]$ satisfying $\omega(\neg\alpha) = 1 - \omega(\alpha)$, $\omega(\alpha \wedge \beta) = \omega(\alpha)\omega(\beta)$, and $\omega(\alpha \vee \beta) = \omega(\alpha) + \omega(\beta) - \omega(\alpha)\omega(\beta)$. A *fuzzy minimum interpretation* ω is a function $\omega : \mathcal{L}(\mathsf{At}) \to [0, 1]$ satisfying $\omega(\neg\alpha) = 1 - \omega(\alpha)$, $\omega(\alpha \wedge \beta) = \min(\omega(\alpha), \omega(\beta))$, and $\omega(\alpha \vee \beta) = \max(\omega(\alpha), \omega(\beta))$. Let Ω_{prod} and $\Omega_{\min}$ be the sets of all fuzzy product interpretations and fuzzy minimum interpretations, respectively.

Definition 18 ([40])**.** The product fuzzy inconsistency measure $\mathcal{I}_{t_{\text{prod}}}^{\text{fuz}}$ is defined as

$$\mathcal{I}_{t_{\text{prod}}}^{\text{fuz}}(\mathcal{K}) = \min\{1 - \omega(\bigwedge \mathcal{K}) \mid \omega \in \Omega_{\text{prod}}\}$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 19 ([40])**.** The minimum-sum fuzzy inconsistency measure $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$ is defined as

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}) = \min\{\sum_{\alpha\in\mathcal{K}}(1-\omega(\alpha)) \mid \omega \in \Omega_{\min}\}$$

for $\mathcal{K} \in \mathbb{K}$.

Definition 20 ([40])**.** The product-sum fuzzy inconsistency measure $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$ is defined as

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}) = \min\{\sum_{\alpha\in\mathcal{K}}(1-\omega(\alpha)) \mid \omega \in \Omega_{\text{prod}}\}$$

for $\mathcal{K} \in \mathbb{K}$.

Note that we do not consider the *minimum fuzzy inconsistency measure*, i. e., the variant of Definition 18 with minimum product interpretation, as this is equivalent to $\mathcal{I}_d$ [40].

A set of maximal consistent subsets $\mathcal{C} \subseteq \mathsf{MC}(\mathcal{K})$ is called an MC-*cover* if

$$\bigcup_{C\in\mathcal{C}} C = K$$

An MC-cover $\mathcal{C}$ is *normal* if no proper subset of $\mathcal{C}$ is an MC-cover. A normal MC-cover is maximal if

$$\lambda(\mathcal{C}) = |\bigcap_{C\in\mathcal{C}} C|$$

is maximal for all normal MC-covers.

Definition 21 ([1])**.** The *MCSC inconsistency measure* $\mathcal{I}_{mcsc} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{mcsc}(\mathcal{K}) = |\mathcal{K}| - \lambda(\mathcal{C})$$

for all $\mathcal{K} \in \mathbb{K}$ and any maximal MC-cover $\mathcal{C}$. If there is no maximal MC-cover we define $\mathcal{I}_{mcsc}(\mathcal{K}) = |\mathcal{K}|$.

Note that the case of the non-existence of a (maximal) MC-cover happens when $\mathcal{K}$ contains an inconsistent formula such as $a \wedge \neg a$. This special case was only implicit in [1].

For a formula ϕ let $\phi[a_1, i_1 \to \psi_1; \ldots, a_k, i_k \to \psi_k]$ denote the formula ϕ where the i_jth occurrence of the proposition a_j is replaced by the formula ψ_j, for all $j = 1, \ldots, k$. For example,

$$(a \wedge b \vee (\neg a \wedge b))[a, 2 \to \top; b, 1 \to \bot] = (a \wedge \bot \vee (\neg \top \wedge b))$$

Definition 22 ([4][2]). The *forgetting-based inconsistency measure* $\mathcal{I}_{\text{forget}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{\text{forget}}(\mathcal{K}) = \min\{k \mid (\bigwedge \mathcal{K})[a_1, i_1 \to \phi_1; \ldots; a_k, i_k \to \phi_k] \not\models \perp, \phi_j \in \{\perp, \top\}\}$$

for all $\mathcal{K} \in \mathbb{K}$.

A set $\{K_1, \ldots, K_n\}$ of pairwise disjoint subsets of $\mathcal{K}$ is called a *conditional independent MUS partition* of $\mathcal{K}$, iff each K_i is inconsistent and $\mathsf{MI}(K_1 \cup \ldots \cup K_n)$ is the disjoint union of all $\mathsf{MI}(K_i)$.

Definition 23 ([18]). The *CC inconsistency measure* $\mathcal{I}_{CC} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{CC}(\mathcal{K}) = \max\{n \mid \{K_1, \ldots, K_n\} \text{ is a conditional independent MUS partition of } \mathcal{K}\}$$

for all $\mathcal{K} \in \mathbb{K}$.

Definition 24 ([17]). The *independent set-based inconsistency measure* $\mathcal{I}_{\text{is}} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$ is defined as

$$\mathcal{I}_{\text{is}}(\mathcal{K}) = \ln |\{K \subseteq \mathsf{MI}(\mathcal{K}) \mid K \text{ consists of pairwise disjoint subsets}\}|$$

for all $\mathcal{K} \in \mathbb{K}$.

Note that [17] did not explicitly define the basis of the logarithm used in the previous definition. As the exact choice only changes the scaling behaviour of the measure, we make it explicit and use the natural logarithm.

The formal definitions of the considered inconsistency measures are summarised in Figure 1.

We conclude this section with a small example illustrating the behavior of the considered inconsistency measures.

Example 1. Let $\mathcal{K}_8$ and $\mathcal{K}_9$ be given as

$$\mathcal{K}_8 = \{a, b \vee c, \neg a \wedge \neg b, d\} \qquad \mathcal{K}_9 = \{a, \neg a, b, \neg b\}$$

Then

$$\begin{aligned} \mathcal{I}_d(\mathcal{K}_8) &= 1 & \mathcal{I}_d(\mathcal{K}_9) &= 1 \\ \mathcal{I}_{\mathsf{MI}}(\mathcal{K}_8) &= 1 & \mathcal{I}_{\mathsf{MI}}(\mathcal{K}_9) &= 2 \\ \mathcal{I}_{\mathsf{MI}^c}(\mathcal{K}_8) &= 1/2 & \mathcal{I}_{\mathsf{MI}^c}(\mathcal{K}_9) &= 1 \\ \mathcal{I}_\eta(\mathcal{K}_8) &= 1/2 & \mathcal{I}_\eta(\mathcal{K}_9) &= 1/2 \\ \mathcal{I}_c(\mathcal{K}_8) &= 1 & \mathcal{I}_c(\mathcal{K}_9) &= 2 \end{aligned}$$

[2]Note that we give a slightly different but equivalent formalization.

$$\mathcal{I}_d(\mathcal{K}) = \begin{cases} 1 & \text{if } \mathcal{K} \models \perp \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = |\mathsf{MI}(\mathcal{K})|$$

$$\mathcal{I}_{\mathsf{MI}^\mathsf{C}}(\mathcal{K}) = \sum_{M \in \mathsf{MI}(\mathcal{K})} \frac{1}{|M|}$$

$$\mathcal{I}_\eta(\mathcal{K}) = 1 - \max\{\xi \mid \exists P \in \mathcal{P}(\mathsf{At}) : \forall \alpha \in \mathcal{K} : P(\alpha) \geq \xi\}$$

$$\mathcal{I}_c(\mathcal{K}) = \min\{|\upsilon^{-1}(B)| \mid \upsilon \models^3 \mathcal{K}\}$$

$$\mathcal{I}_{mc}(\mathcal{K}) = |\mathsf{MC}(\mathcal{K})| + |\mathsf{SC}(\mathcal{K})| - 1$$

$$\mathcal{I}_p(\mathcal{K}) = |\bigcup_{M \in \mathsf{MI}(\mathcal{K})} M|$$

$$\mathcal{I}_{hs}(\mathcal{K}) = \min\{|H| \mid H \text{ is a hitting set of } \mathcal{K}\} - 1$$

$$\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}) = \min\{\sum_{\alpha \in \mathcal{K}} d_{\text{d}}(\mathsf{Mod}(\alpha), \omega) \mid \omega \in \Omega(\mathsf{At})\}$$

$$\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}) = \min\{\max_{\alpha \in \mathcal{K}} d_{\text{d}}(\mathsf{Mod}(\alpha), \omega) \mid \omega \in \Omega(\mathsf{At})\}$$

$$\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}) = \min\{|\{\alpha \in \mathcal{K} \mid d_{\text{d}}(\mathsf{Mod}(\alpha), \omega) > 0\}| \mid \omega \in \Omega(\mathsf{At})\}$$

$$\mathcal{I}_{D_f}(\mathcal{K}) = 1 - \Pi_{i=1}^{|\mathcal{K}|}(1 - R_i(\mathcal{K})/i)$$

$$\mathcal{I}_{P_m}(\mathcal{K}) = \sum_{a \in \mathsf{At}} |P_m^{\mathcal{K}}(a)| \cdot |P_m^{\mathcal{K}}(\neg a)|$$

$$\mathcal{I}_{mv}(\mathcal{K}) = \frac{|\bigcup_{M \in \mathsf{MI}(\mathcal{K})} \mathsf{At}(M)|}{|\mathsf{At}(\mathcal{K})|}$$

$$\mathcal{I}_{nc}(\mathcal{K}) = |\mathcal{K}| - \max\{n \mid \forall \mathcal{K}' \subseteq \mathcal{K} : |\mathcal{K}'| = n \Rightarrow \mathcal{K}' \not\models \perp\}$$

$$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}) = \min\{1 - \omega(\bigwedge \mathcal{K}) \mid \omega \in \Omega_{\text{prod}}\}$$

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}) = \min\{\sum_{\alpha \in \mathcal{K}} (1 - \omega(\alpha)) \mid \omega \in \Omega_{\min}\}$$

$$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}) = \min\{\sum_{\alpha \in \mathcal{K}} (1 - \omega(\alpha)) \mid \omega \in \Omega_{\text{prod}}\}$$

$$\mathcal{I}_{mcsc}(\mathcal{K}) = |\mathcal{K}| - \lambda(\mathcal{C})$$

$$\mathcal{I}_{\text{forget}}(\mathcal{K}) = \min\{k \mid (\bigwedge \mathcal{K})[a_1, i_1 \to \phi_1; \ldots; a_k, i_k \to \phi_k] \not\models \perp, \phi_j \in \{\perp, \top\}\}$$

$$\mathcal{I}_{CC}(\mathcal{K}) = \max\{n \mid \{K_1, \ldots, K_n\} \text{ is a CI partition of } \mathcal{K}\}$$

$$\mathcal{I}_{\text{is}}(\mathcal{K}) = \ln |\{K \subseteq \mathsf{MI}(\mathcal{K}) \mid K \text{ consists of pairwise disjoint subsets}\}|$$

Figure 1: Definitions of the considered inconsistency measures

$\mathcal{I}_{mc}(\mathcal{K}_8) = 1$	$\mathcal{I}_{mc}(\mathcal{K}_9) = 3$
$\mathcal{I}_{p}(\mathcal{K}_8) = 2$	$\mathcal{I}_{p}(\mathcal{K}_9) = 4$
$\mathcal{I}_{hs}(\mathcal{K}_8) = 1$	$\mathcal{I}_{hs}(\mathcal{K}_9) = 1$
$\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}_8) = 1$	$\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K}_9) = 2$
$\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}_8) = 1$	$\mathcal{I}^{\max}_{\text{dalal}}(\mathcal{K}_9) = 1$
$\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}_8) = 1$	$\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K}_9) = 2$
$\mathcal{I}_{D_f}(\mathcal{K}_8) \approx 0.083$	$\mathcal{I}_{D_f}(\mathcal{K}_9) \approx 0.167$
$\mathcal{I}_{P_m}(\mathcal{K}_8) = 1$	$\mathcal{I}_{P_m}(\mathcal{K}_9) = 2$
$\mathcal{I}_{mv}(\mathcal{K}_8) = 1/2$	$\mathcal{I}_{mv}(\mathcal{K}_9) = 1$
$\mathcal{I}_{nc}(\mathcal{K}_8) = 3$	$\mathcal{I}_{nc}(\mathcal{K}_9) = 3$
$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}_8) = 0.75$	$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}(\mathcal{K}_9) = 0.9375$
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}_8) = 1$	$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}(\mathcal{K}_9) = 2$
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}_8) = 1$	$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}(\mathcal{K}_9) = 2$
$\mathcal{I}_{mcsc}(\mathcal{K}_8) = 2$	$\mathcal{I}_{mcsc}(\mathcal{K}_9) = 4$
$\mathcal{I}_{\text{forget}}(\mathcal{K}_8) = 1$	$\mathcal{I}_{\text{forget}}(\mathcal{K}_9) = 2$
$\mathcal{I}_{CC}(\mathcal{K}_8) = 1$	$\mathcal{I}_{CC}(\mathcal{K}_9) = 2$
$\mathcal{I}_{\text{is}}(\mathcal{K}_8) \approx 0.693$	$\mathcal{I}_{\text{is}}(\mathcal{K}_9) \approx 1.386$

4 Rationality Postulates

In the following, we recall 18 rationality postulates that have been proposed in the literature [14, 35, 16, 31, 30, 37, 3]. A previous survey of rationality postulates can be found in [41].

The first set of rationality postulates has been proposed in [14] in order to provide a definition of a *basic inconsistency measure*. In order to state these postulates we need one further definition.

Definition 25. A formula $\alpha \in \mathcal{K}$ is called a *free formula* if $\alpha \notin \bigcup \mathsf{MI}(\mathcal{K})$. Let $\mathsf{Free}(\mathcal{K})$ be the set of all free formulas of $\mathcal{K}$.

In other words, a free formula is basically a formula that is not directly participating in any derivation of a contradiction. Using this definition and the concepts already introduced before, the first five rationality postulates of [14] can be stated as follows. For the remainder of this section, let $\mathcal{I}$ be any function $\mathcal{I} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$, $\mathcal{K}, \mathcal{K}' \in \mathbb{K}$, and $\alpha, \beta \in \mathcal{L}(\mathsf{At})$.

Consistency **(CO)** $\mathcal{I}(\mathcal{K}) = 0$ if and only if $\mathcal{K}$ is consistent

Normalization **(NO)** $0 \leq \mathcal{I}(\mathcal{K}) \leq 1$

Monotony **(MO)** If $\mathcal{K} \subseteq \mathcal{K}'$ then $\mathcal{I}(\mathcal{K}) \leq \mathcal{I}(\mathcal{K}')$

Free-formula independence **(IN)** If $\alpha \in \mathsf{Free}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$

Dominance **(DO)** If $\alpha \not\models \perp$ and $\alpha \models \beta$ then $\mathcal{I}(\mathcal{K} \cup \{\alpha\}) \geq \mathcal{I}(\mathcal{K} \cup \{\beta\})$

The first postulate, CO, requires that consistent knowledge bases receive the minimal inconsistency value zero and every inconsistent knowledge base have a strictly positive inconsistency value. This postulate is actually the only generally accepted postulate and describes the minimal requirement for an inconsistency measure. An inconsistency measure $\mathcal{I}$ that satisfies CO does not distinguish between consistent knowledge bases and can, at least, distinguish inconsistent knowledge bases from consistent ones.

The postulate NO states that the inconsistency value is always in the unit interval, thus allowing inconsistency values to be comparable even if knowledge bases are of different sizes. In later works, this postulate is usually regarded as an optional feature, because many measures tend to assess inconsistency *absolutely* and not *relatively*. The distinction between these two points of view was already made in [9], but a thorough investigation of the implications for taking either view on the validity of other postulates has still to be made.

MO requires that adding formulas to the knowledge base cannot decrease the inconsistency value. Besides CO this is the least disputed postulate and most inconsistency measures do satisfy it (see below).

IN states that removing free formulas from the knowledge base should not change the inconsistency value. The motivation here is that free formulas do not participate in inconsistencies and should not contribute to having a certain inconsistency value.

DO says that substituting a consistent formula α by a weaker formula β should not increase the inconsistency value. Here, as β carries less information than α there should be less opportunities for inconsistencies to occur.[3]

The set of postulates was extended in [35] for the case of inconsistency measurement in probabilistic logics. However, we can state these postulates also for propositional logic.

Definition 26. A formula $\alpha \in \mathcal{K}$ is called a *safe formula* if it is consistent and $\mathsf{At}(\alpha) \cap \mathsf{At}(\mathcal{K} \setminus \{\alpha\}) = \emptyset$. Let $\mathsf{Safe}(\mathcal{K})$ be the set of all safe formulas of $\mathcal{K}$.

A formula is safe if its signature is disjoint from the signature of the rest of the knowledge base, cf. the concept of language splitting in belief revision [34, 24]. Every safe formula is also a free formula [35].

[3] A weaker version of DO has also been discussed in [2, 6]. In this version the additional condition $\alpha \notin \mathcal{K}$ is added to the postulate. The special case $\alpha \in \mathcal{K}$ is usually the reason that measures do not satisfy the original version of DO; we leave a thorough study of this weaker version for future work.

Safe-formula independence **(SI)** If $\alpha \in \mathsf{Safe}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$

Super-Additivity **(SA)** If $\mathcal{K} \cap \mathcal{K}' = \emptyset$ then $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') \geq \mathcal{I}(\mathcal{K}) + \mathcal{I}(\mathcal{K}')$

Penalty **(PY)** If $\alpha \notin \mathsf{Free}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) > \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$

The postulate SI requires that removing isolated formulas from a knowledge base cannot change the inconsistency value. This postulate is a weakening of IN, i. e., if a measure $\mathcal{I}$ satisfies IN it also satisfies SI, cf. [35, 41] and Theorem 1.

SA is a strengthening of MO [35] and requires that the sum of the inconsistency values of two disjoint knowledge bases not be larger than the inconsistency value of the joint knowledge base.

PY is the complementary postulate to IN and states that adding a formula participating in an inconsistency must have a positive impact on the inconsistency value.

The following two postulates have been first used in [16]:

MI-*separability* **(MI)** If $\mathsf{MI}(\mathcal{K} \cup \mathcal{K}') = \mathsf{MI}(\mathcal{K}) \cup \mathsf{MI}(\mathcal{K}')$ and $\mathsf{MI}(\mathcal{K}) \cap \mathsf{MI}(\mathcal{K}') = \emptyset$ then $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') = \mathcal{I}(\mathcal{K}) + \mathcal{I}(\mathcal{K}')$

MI-*normalization* **(MN)** If $M \in \mathsf{MI}(\mathcal{K})$ then $\mathcal{I}(M) = 1$

MI focuses particularly on the role of minimal inconsistent subsets in the determination of the inconsistency value. It states that the sum of the inconsistency values of two knowledge bases that have "non-interfering" sets of minimal inconsistent subsets should be the same as the inconsistency value of their union.

MN demands that a minimal inconsistent subset is the atomic unit for measuring inconsistency by requiring that the inconsistency value of any minimal inconsistent subset be one.

The following postulates have been proposed in [30] to further define the role of minimal inconsistent subsets in measuring inconsistency[4]:

Attenuation **(AT)** $M, M' \in \mathsf{MI}(\mathcal{K})$ and $|M| > |M'|$ implies $\mathcal{I}(M) < \mathcal{I}(M')$

Equal Conflict **(EC)** $M, M' \in \mathsf{MI}(\mathcal{K})$ and $|M| = |M'|$ implies $\mathcal{I}(M) = \mathcal{I}(M')$

Almost Consistency **(AC)** Let $M_1, M_2, \ldots$ be a sequence of minimal inconsistent sets M_i with $\lim_{i \to \infty} |M_i| = \infty$, then $\lim_{i \to \infty} \mathcal{I}(M_i) = 0$

[4]Note that in the previous study on compliance of rationality postulates [41] the postulates AT and EC were stated in a slightly different way, we give here the original definitions.

The postulate AT states that minimal inconsistent sets of smaller size should have a larger inconsistency value. The motivation of this postulate stems from the *lottery paradox*[5] [25].

The postulate EC is the counterpart of AT and requires minimal inconsistent subsets of the same size to have the same inconsistency value.

AC considers the inconsistency values on arbitrarily large minimal inconsistent subsets in the limit and requires this to be zero.

The following postulates are from [31].

Contradiction **(CD)** $\mathcal{I}(\mathcal{K}) = 1$ if and only if for all $\emptyset \neq \mathcal{K}' \subseteq \mathcal{K}$, $\mathcal{K}' \models \perp$

Free Formula Dilution **(FD)** If $\alpha \in \mathsf{Free}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) \geq \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$

CD is meant as an extension of NO and states that a knowledge base is maximally inconsistent if all non-empty subsets are inconsistent. Note that CD only makes sense if NO is satisfied as well. FD has been introduced to serve as a weaker version of IN for normalised measures, i. e., measures satisfying NO. For those, it may be the case that adding free formulas decreases the inconsistency value as they measure a "relative" amount of inconsistency. We do not consider here the property *Monotony w.r.t. Conflict Ratio* from [31] as it is too specifically tailored for the measure $\mathcal{I}_{D_f}$.

The following property has been mentioned independently in [36] and [10]:

Irrelevance of Syntax **(SY)** If $\mathcal{K} \equiv_b \mathcal{K}'$ then $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K}')$

SY states that knowledge bases with pairwise equivalent formulas should receive the same inconsistency value.

In [3] a series of further postulates have been discussed. For our current study, we only consider the following two:

Exchange **(EX)** If $\mathcal{K}' \not\models \perp$ and $\mathcal{K}' \equiv \mathcal{K}''$ then $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') = \mathcal{I}(\mathcal{K} \cup \mathcal{K}'')$

Adjunction Invariance **(AI)** $\mathcal{I}(\mathcal{K} \cup \{\alpha, \beta\}) = \mathcal{I}(\mathcal{K} \cup \{\alpha \wedge \beta\})$

EX is similar in spirit to SY and demands that exchanging consistent parts of the knowledge base with equivalent ones should not change the inconsistency value.

AI demands that the set notation of knowledge bases should be equivalent to the conjunction of its formulas in terms of inconsistency values. In difference to EX note that AI has no precondition on the consistency of the considered formulas.

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[5] Consider a lottery of n tickets and let a_i be the proposition that ticket i, $i = 1, \ldots, n$ will win. It is known that exactly one ticket will win ($a_1 \vee \ldots \vee a_n$) but each ticket owner assumes that his ticket will not win ($\neg a_i$, $i = 1, \ldots, n$). For $n = 1000$ it is reasonable for each ticket owner to believe that he will not win but for e. g., $n = 2$ it is not. Therefore larger minimal inconsistent subsets can be regarded less inconsistent than smaller ones.

Note that not all postulates are independent and that some are incompatible. Some relationships are summarised in the following theorem, see [41] for proofs of items 1–8, [2] for proofs of items 9 and 10, proofs of items 11 and 12 are trivial and omitted. In the theorem, a statement "A implies B" is meant to be read as "if a measure satisfies A then it satisfies B"; a statement "A_1, ..., A_n are incompatible" means "there is no measure satisfying A_1, ..., A_n at the same time".

Theorem 1.

1. IN implies SI
2. IN implies FD
3. SA implies MO
4. MN and AC are incompatible
5. MN and CD are incompatible
6. MO implies FD
7. MN, MI, and NO are incompatible
8. MN, SA, and NO are incompatible
9. CO, DO, and SA are incompatible
10. CO, DO, and MI are incompatible
11. MN implies EC
12. MN and AT are incompatible

See also [3, 2] for some more detailed discussions.

Tables 2 and 3 give the complete picture on which inconsistency measure satisfies (✓) or violates (✗) the previously discussed rationality postulates. Some of these results have been shown before in [23, 15, 16, 31, 10, 46, 37, 12, 21, 18, 39, 4, 41, 40][6], marked correspondingly in Tables 2 and 3. The proofs and counterexamples of the remaining statements are given in the appendix. Note that in [46] it has been shown that $\mathcal{I}_{mv}$ satisfies restricted versions of MO and IN where only formulas are considered that do not use fresh propositions. Some results reported here correct previous statements. In particular, $\mathcal{I}_{P_m}$ does not satisfy CO as claimed in [21], $\mathcal{I}_{nc}$ does not satisfy IN, SI, and DO as claimed in [41], and both $\mathcal{I}^{\Sigma}_{\text{dalal}}$ and $\mathcal{I}^{\text{hit}}_{\text{dalal}}$ do not satisfy DO as claimed in [12]. Due to a different phrasing of the postulates AT and EC in [41] compared

[6]Note that proofs of [37] are for propositional probabilistic logic. As this is a generalization of propositional logic, the results apply here as well.

$\mathcal{I}$	CO	NO	MO	IN	DO	SI	SA	PY	MI	MN
$\mathcal{I}_d$	✓[16]	✓[41]	✓[16]	✓[16]	✓[16]	✓[37]	✗[37]	✗[37]	✗[37]	✓[41]
$\mathcal{I}_{\mathsf{MI}}$	✓[15]	✗[41]	✓[15]	✓[15]	✗[31]	✓[37]	✓[37]	✓[37]	✓[16]	✓[16]
$\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$	✓[10]	✗[37]	✓[10]	✓[10]	✗[41]	✓[37]	✓[37]	✓[37]	✓[37]	✗[41]
$\mathcal{I}_\eta$	✓[23]	✓[23]	✓[37]	✓[37]	✓[41]	✓[37]	✗[37]	✗[37]	✗[37]	✗[41]
$\mathcal{I}_c$	✓[10]	✗[41]	✓[10]	✓[10]	✓[41]	✓[41]	✗[41]	✗[41]	✗[41]	✗[41]
$\mathcal{I}_{mc}$	✓[10]	✗[41]	✓[10]	✓[10]	✗[41]	✓[41]	✗[41]	✗[41]	✗[18]	✗[41]
$\mathcal{I}_p$	✓[10]	✗[41]	✓[10]	✓[10]	✗[41]	✓[41]	✓[41]	✓[41]	✗[41]	✗[41]
$\mathcal{I}_{hs}$	✓[39]	✗[41]	✓[39]	✓[39]	✓[39]	✓[39]	✗[39]	✗[41]	✗[39]	✗[41]
$\mathcal{I}^{\Sigma}_{\text{dalal}}$	✓[12]	✗[41]	✓[12]	✓[12]	✗	✓[41]	✓[41]	✗[41]	✗[41]	✗[41]
$\mathcal{I}^{\max}_{\text{dalal}}$	✓[12]	✗[41]	✓[12]	✓[12]	✓[12]	✓[41]	✗[41]	✗[41]	✗[41]	✗[41]
$\mathcal{I}^{\text{hit}}_{\text{dalal}}$	✓[12]	✗[41]	✓[12]	✓[12]	✗	✓[41]	✓[41]	✗[41]	✗[41]	✓[41]
$\mathcal{I}_{D_f}$	✓[31]	✓[31]	✗[41]	✗[41]	✗[41]	✗[41]	✗[41]	✗[41]	✗[41]	✗[41]
$\mathcal{I}_{P_m}$	✗	✗[41]	✓[21]	✗[21]	✗[21]	✓[41]	✓[41]	✓[41]	✗[41]	✗[41]
$\mathcal{I}_{mv}$	✓[46]	✓[41]	✗[41]	✗[41]	✗[41]	✗[41]	✗[41]	✗[41]	✗[41]	✓[41]
$\mathcal{I}_{nc}$	✓[41]	✗[41]	✓[41]	✗	✗	✗	✓[41]	✓[41]	✗[41]	✓[41]
$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$	✓[40]	✓[40]	✓[40]	✗[40]	✗[40]	✓[40]	✗[40]	✗[40]	✗[40]	✗[40]
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$	✓[40]	✗[40]	✓[40]	✗[40]	✓[40]	✓[40]	✓[40]	✗[40]	✗[40]	✗[40]
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$	✓[40]	✗[40]	✓[40]	✗[40]	✗[40]	✓[40]	✓[40]	✗[40]	✗[40]	✗[40]
$\mathcal{I}_{mcsc}$	✓[1]	✗	✓[1]	✓[1]	✗[1]	✓	✓[1]	✗	✗[1]	✗
$\mathcal{I}_{\text{forget}}$	✓[4]	✗	✓[4]	✓[4]	✗[4]	✓	✓	✗	✗	✗
$\mathcal{I}_{CC}$	✓	✗	✓[18]	✓	✗	✓	✗[20]	✗	✓[18]	✓
$\mathcal{I}_{\text{is}}$	✓[17]	✗	✓[17]	✓[17]	✗	✓	✓	✓	✓[17]	✓[17]

Table 2: Compliance of inconsistency measures with rationality postulates CO, NO, MO, IN, DO, SI, SA, PY, MI, and MN; previous results are indicated by a super-scripted reference of the original work (some of the results have been shown in multiple publications, only the first occurrence is cited)

to their original definitions in [30], we also corrected some results pertaining to these. See Appendix 7 for the corresponding proofs and counterexamples.

The only rationality postulate that almost all considered measures agree upon is CO, which is not surprising as it captures the minimal requirement for any inconsistency measure.[7] Most measures also satisfy MO, which is also the least disputed in the literature. The only cases where MO fails is usually when NO is satisfied, cf. $\mathcal{I}_{D_f}$ and $\mathcal{I}_{mv}$. However, note that MO and NO are not generally incompatible as e. g. $\mathcal{I}_\eta$ satisfies both. Some other postulates are violated by most of the considered inconsistency measures, in particular if they address a very specific feature. For example, CD is motivated by the measure $\mathcal{I}_{D_f}$—which is also the only one satisfying it—and can be seen as the counterpart to CO as it describes a concept of *maximal inconsistency*. Of course, requiring that a *maximally inconsistent* knowledge base receive the maximal

[7]The fact that $\mathcal{I}_{P_m}$ violates CO is also unintentional as the original paper [21] falsely claimed that CO is satisfied

$\mathcal{I}$	AT	EC	AC	CD	FD	SY	EX	AI
$\mathcal{I}_d$	✗	✓	✗[41]	✗[41]	✓[41]	✓[37]	✓[41]	✓[41]
$\mathcal{I}_{\mathsf{MI}}$	✗	✗	✗[41]	✗[41]	✓[41]	✓[10]	✗[41]	✗[41]
$\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$	✓[41]	✓[41]	✓[41]	✗[41]	✓[41]	✓[10]	✗[41]	✗[41]
$\mathcal{I}_\eta$	✓[41]	✓[41]	✓[41]	✗[41]	✓[41]	✓[37]	✗[41]	✗[41]
$\mathcal{I}_c$	✗[41]	✗[41]	✗[41]	✗[41]	✓[41]	✗[10]	✓[41]	✓[41]
$\mathcal{I}_{mc}$	✗[41]	✓	✗[41]	✗[41]	✓[41]	✓[10]	✗[41]	✗[41]
$\mathcal{I}_p$	✗[41]	✓[41]	✗[41]	✗[41]	✓[41]	✓[10]	✗[41]	✗[41]
$\mathcal{I}_{hs}$	✗	✓	✗[41]	✗[41]	✓[41]	✓[39]	✗[41]	✗[41]
$\mathcal{I}^{\Sigma}_{\text{dalal}}$	✗[41]	✗[41]	✗[41]	✗[41]	✓[41]	✓[41]	✗[41]	✗[41]
$\mathcal{I}^{\max}_{\text{dalal}}$	✗[41]	✗[41]	✗[41]	✗[41]	✓[41]	✓[41]	✗[41]	✗[41]
$\mathcal{I}^{\text{hit}}_{\text{dalal}}$	✗	✓	✗[41]	✗[41]	✓[41]	✓[41]	✗[41]	✗[41]
$\mathcal{I}_{D_f}$	✓[31]	✓[41]	✓[31]	✓[31]	✓[31]	✓[41]	✗[41]	✗[41]
$\mathcal{I}_{P_m}$	✗[41]	✗[41]	✗[41]	✗[41]	✓[41]	✗[41]	✗[41]	✗[41]
$\mathcal{I}_{mv}$	✗	✓	✗[41]	✗[41]	✗[41]	✗[41]	✗[41]	✗[41]
$\mathcal{I}_{nc}$	✗	✓	✗[41]	✗[41]	✓[41]	✓[41]	✗[41]	✗[41]
$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$	✗[40]	✗[40]	✗[40]	✗[40]	✓[40]	✗[40]	✗[40]	✓[40]
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{min}}}$	✗[40]	✓	✗[40]	✗[40]	✓[40]	✓[40]	✗[40]	✗[40]
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$	✗[40]	✗[40]	✗[40]	✗[40]	✓[40]	✗[40]	✗[40]	✗[40]
$\mathcal{I}_{mcsc}$	✗	✓	✗	✗	✓	✓	✗	✗
$\mathcal{I}_{\text{forget}}$	✗	✗	✗	✗	✓	✗	✗[4]	✓[4]
$\mathcal{I}_{CC}$	✗	✓	✗	✗	✓	✓	✗	✗
$\mathcal{I}_{\text{is}}$	✗	✓	✗	✗	✓	✓	✗	✗

Table 3: Compliance of inconsistency measures with rationality postulates AT, EC, AC, CD, FD, SY, EX, and AI; previous results are indicated by a superscripted reference of the original work (some of the results have been shown in multiple publications, only the first occurrence is cited)

possible inconsistency value is a desirable property. The specific instance of this requirement in CD, i. e., that *maximal inconsistency* is defined by not having non-empty consistent subsets and that the maximal value is 1, is very specific to $\mathcal{I}_{D_f}$. The value 1 only makes sense when the measure is normalized, so that 1 is indeed the maximal possible value. Moreover, the definition of *maximal inconsistency* requires some more investigation.

One important thing to note from the results shown in Tables 2 and 3, is that all investigated inconsistency measures satisfy different sets of postulates. More precisely, there are no two inconsistency measures $\mathcal{I}$ and $\mathcal{I}'$ that satisfy and violate the exact same set of postulates. This also means that we can find knowledge bases $\mathcal{K}, \mathcal{K}'$ such that $\mathcal{I}(\mathcal{K}) < \mathcal{I}(\mathcal{K}')$ and $\mathcal{I}'(\mathcal{K}) \geq \mathcal{I}'(\mathcal{K}')$, meaning that all considered inconsistency measures are essentially different.[8]

[8] An earlier observation regarding a subset of the investigated measures has been made in [10].

5 Expressivity

The drastic inconsistency measure $\mathcal{I}_d$ (see Figure 1) is usually considered as a very naive baseline approach for inconsistency measurement. Surprisingly, this measure already satisfies many rationality postulates, cf. Tables 2 and 3. What sets it apart from other more sophisticated inconsistency measures is that it cannot differentiate between different inconsistent knowledge bases. However, this demand is exactly what inconsistency measures are supposed to satisfy. To address this issue, the work [38] initiated the analysis of the *expressivity* of inconsistency measures. With expressivity of inconsistency measures we here mean the number of different values an inconsistency measure can attain.

Example 2. Consider the knowledge bases $\mathcal{K}_{10}$ and $\mathcal{K}_{11}$ defined via

$$\mathcal{K}_{10} = \{a, b, \neg a \vee \neg b, c, d, \neg c \vee \neg d\}$$
$$\mathcal{K}_{11} = \{a, \neg a, b, \neg b\}$$

Both knowledge bases contain two minimal inconsistent subsets and, thus, $\mathcal{I}_{\mathsf{MI}}$ is not able to differentiate their severity of inconsistency (recall that $\mathcal{I}_{\mathsf{MI}}$ takes the number of minimal inconsistent subsets as the inconsistency values)

$$\mathcal{I}_{\mathsf{MI}}(\mathcal{K}_{10}) = \mathcal{I}_{\mathsf{MI}}(\mathcal{K}_{11}) = 2$$

On the other hand, $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ does distinguish $\mathcal{K}_{10}$ and $\mathcal{K}_{11}$ (recall that $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ sums the reciprocal sizes of all minimal inconsistent subsets)

$$\mathcal{I}_{\mathsf{MI}^\mathsf{C}}(\mathcal{K}_{10}) = 2/3 \qquad\qquad \mathcal{I}_{\mathsf{MI}^\mathsf{C}}(\mathcal{K}_{11}) = 1$$

Therefore, $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ can be regarded as more *expressive* than $\mathcal{I}_{\mathsf{MI}}$ wrt. $\mathcal{K}_{10}$ and $\mathcal{K}_{11}$

Example 3. Consider the family of knowledge bases $\mathcal{K}_{12}^i$ (for $i \in \mathbb{N}$)

$$\mathcal{K}_{12}^i = \{a_1 \wedge \ldots \wedge a_i, \neg a_1 \wedge \ldots \wedge \neg a_i\}$$

Observe that $\mathcal{K}_{12}^i$ contains one minimal inconsistent subset (independently of i) and therefore

$$\mathcal{I}_{\mathsf{MI}}(\mathcal{K}_{12}^i) = 1$$

for all $i \in \mathbb{N}$. However, $\mathcal{I}_c$ is able to distinguish every single member of the family (recall that, roughly, $\mathcal{I}_c$ counts the number of propositions which are involved in conflicts)

$$\mathcal{I}_c(\mathcal{K}_{12}^i) = i$$

for $i \in \mathbb{N}$. Therefore, $\mathcal{I}_c$ can be regarded as more *expressive* than $\mathcal{I}_{\mathsf{MI}}$ wrt. $\mathcal{K}_{12}^i$.

In the following, we recall the framework of [38] and investigate the expressivity of inconsistency measures along four different dimensions of subclasses of knowledge bases.

Definition 27. Let ϕ be a formula. The *length* $\texttt{len}(\phi)$ of ϕ is recursively defined as

$$\texttt{len}(\phi) = \begin{cases} 1 & \text{if } \phi \in \mathsf{At} \\ 1 + \texttt{len}(\phi') & \text{if } \phi = \neg\phi' \\ 1 + \texttt{len}(\phi_1) + \texttt{len}(\phi_2) & \text{if } \phi = \phi_1 \wedge \phi_2 \\ 1 + \texttt{len}(\phi_1) + \texttt{len}(\phi_2) & \text{if } \phi = \phi_1 \vee \phi_2 \end{cases}$$

In other words $\texttt{len}(\phi)$ is the number of connectives plus the number of occurrences of atom in ϕ. Furthermore, we treat $\phi_1 \to \phi_2$ as an abbreviation of $\neg\phi_1 \vee \phi_2$ and therefore $\texttt{len}(\phi_1 \to \phi_2) = 2 + \texttt{len}(\phi_1) + \texttt{len}(\phi_2)$.

Definition 28. Define the following subclasses of the set of all knowledge bases $\mathbb{K}$:

$$\begin{aligned} \mathbb{K}^v(n) &= \{\mathcal{K} \in \mathbb{K} \mid |\mathsf{At}(\mathcal{K})| \leq n\} \\ \mathbb{K}^f(n) &= \{\mathcal{K} \in \mathbb{K} \mid |\mathcal{K}| \leq n\} \\ \mathbb{K}^l(n) &= \{\mathcal{K} \in \mathbb{K} \mid \forall \phi \in \mathcal{K} : \texttt{len}(\phi) \leq n\} \\ \mathbb{K}^p(n) &= \{\mathcal{K} \in \mathbb{K} \mid \forall \phi \in \mathcal{K} : |\mathsf{At}(\phi)| \leq n\} \end{aligned}$$

In other words, $\mathbb{K}^v(n)$ is the set of all knowledge bases that mention at most n different propositions; $\mathbb{K}^f(n)$ is the set of all knowledge bases that contain at most n formulas; $\mathbb{K}^l(n)$ is the set of all knowledge bases that contain only formulas with maximal length n; and $\mathbb{K}^p(n)$ is the set of all knowledge bases that contain only formulas that mention at most n different propositions each. The motivation for considering these particular subclasses of knowledge bases is that each of them considers a different aspect of the size of a knowledge base. As a syntactical object, a knowledge base is a set of formulas, and both the number of formulas (considered by the class $\mathbb{K}^f(n)$) and the length of each formula ($\mathbb{K}^l(n)$) are the essential parameters that define its size. From a semantical point of view, the number of propositions appearing in each formula ($\mathbb{K}^p(n)$) and in the complete knowledge base ($\mathbb{K}^v(n)$) define the scope of the knowledge. Larger numbers for both of them also indicate larger scope and thus greater size. Inconsistency measures should adhere to the size of the knowledge base in terms of their expressivity. For example, the number of possible inconsistency values of a particular measure should not decrease when moving from a set $\mathbb{K}^v(n)$ to a set $\mathbb{K}^v(n')$ with $n' > n$, as knowledge bases with n' formulas should provide a larger variety in terms of inconsistency than knowledge bases of size n. Indeed, this property is true for all considered measures as $\mathbb{K}^v(n) \subseteq \mathbb{K}^v(n')$ (the same holds for all classes above).

The aim of an expressivity analysis is to investigate the number of different values that a specific inconsistency measure can attain on different subclasses of knowledge bases. This idea can be formalised by *expressivity characteristics* [38].

Definition 29. Let $\mathcal{I}$ be an inconsistency measure and $n > 0$. Let $\alpha \in \{v, f, l, p\}$. The *$\alpha$-characteristic* $\mathcal{C}^\alpha(\mathcal{I}, n)$ of $\mathcal{I}$ wrt. n is defined as

$$\mathcal{C}^\alpha(\mathcal{I}, n) = |\{\mathcal{I}(\mathcal{K}) \mid \mathcal{K} \in \mathbb{K}^\alpha(n)\}|$$

In other words, $\mathcal{C}^\alpha(\mathcal{I}, n)$ is the number of different inconsistency values $\mathcal{I}$ assigns to knowledge bases from $\mathbb{K}^\alpha(n)$.

Table 4 shows the expressivity characteristics for all measures considered in this paper. Proofs pertaining to measures $\mathcal{I}_d$, $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{\mathsf{MI^C}}$, $\mathcal{I}_\eta$, $\mathcal{I}_c$, $\mathcal{I}_{mc}$, $\mathcal{I}_p$, $\mathcal{I}_{hs}$, $\mathcal{I}^{\Sigma}_{\text{dalal}}$, $\mathcal{I}^{\max}_{\text{dalal}}$, $\mathcal{I}^{\text{hit}}_{\text{dalal}}$, $\mathcal{I}_{D_f}$, $\mathcal{I}_{P_m}$, $\mathcal{I}_{mv}$, and $\mathcal{I}_{nc}$ can be found in [38]. Proofs pertaining to measures $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$, $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$, and $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$ can be found in [40]. The remaining proofs are given in the appendix.

The evaluation shows that inconsistency measures behave quite differently wrt. expressivity. First, the analysis clearly shows that the inconsistency measure $\mathcal{I}_d$ is indeed a poor inconsistency measure as it has a constant expressivity value in all four considered dimensions. Second, one can categorise measures into different clusters pertaining to each expressivity characteristic and with significant differences between the values in multiple order of magnitudes. For example, for $\mathcal{C}^v$ we have one measure with constant expressivity value ($\mathcal{I}_d$) and several with an expressivity value of linear size ($\mathcal{I}_c$, $\mathcal{I}^{\max}_{\text{dalal}}$, and $\mathcal{I}_{mv}$). Next, there are two measures with an expressivity value of exponential size ($\mathcal{I}_\eta$ and $\mathcal{I}_{hs}$) and, finally, several measures with infinite expressivity values. This gives us a clear superiority relation wrt. each concrete expressivity characteristic. Third, evaluating expressivity depends highly on the characteristic. As one can see from Table 4, the rankings on expressivity induced by the characteristics $\mathcal{C}^v$ and $\mathcal{C}^f$ are reversed in some places. Consider e. g., the measures $\mathcal{I}_p$ and $\mathcal{I}_c$. The measure $\mathcal{I}_p$ has a rather low expressivity value wrt. $\mathcal{C}^f$ but a high value wrt. $\mathcal{C}^v$. Conversely, $\mathcal{I}_c$ has a high expressivity value wrt. $\mathcal{C}^f$ but a rather low value wrt. $\mathcal{C}^v$. Similar observations can be made for other measures. The reason for this is that the expressivity characteristics $\mathcal{C}^f$ and $\mathcal{C}^v$ provide a means to differentiate between so-called syntactic measures and semantical measures, cf. [13]. This categorisation aims at classifying measures on whether they operate on the formula level (syntactic measures) or on the proposition level (semantic measures). While the original definition of syntactical and semantical measure is rather informal, the expressivity characteristics $\mathcal{C}^f$ and $\mathcal{C}^v$ make this distinction more precise. In particular, $\mathcal{C}^v$ measures how susceptible a measure is when the vocabulary, i. e. the semantical side of the knowledge base, is restricted. Semantical measures such as $\mathcal{I}_c$ have a low expressivity when the vocabulary is restricted. On the other hand, $\mathcal{C}^f$ measures how susceptible a measure is when the number of formulas, i. e., the syntactical part, is restricted. Syntactical measures such as $\mathcal{I}_p$ have a low expressivity in this case.

There are some measures ($\mathcal{I}^{\Sigma}_{\text{dalal}}$, $\mathcal{I}_{P_m}$, $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$, $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$, $\mathcal{I}_{\text{forget}}$) that have infinite expressivity values in all considered dimensions. Just from the point of view of expressivity, these measures seem to be good candidates for "good" measures. However, as the previous section already discussed, the satisfaction of certain rationality postulates may be of more importance than expressivity.

6 Computational Complexity

The final evaluation criterion we consider is *computational complexity* [32]. Following [44, 43], we consider the following three decision problems one can consider for inconsistency measures. Let $\mathcal{I}$ be some inconsistency measure.

$\textsc{Exact}_{\mathcal{I}}$	**Input**:	$\mathcal{K} \in \mathbb{K}$, $x \in \mathbb{R}^{\infty}_{\geq 0}$
	Output:	TRUE iff $\mathcal{I}(\mathcal{K}) = x$
$\textsc{Upper}_{\mathcal{I}}$	**Input**:	$\mathcal{K} \in \mathbb{K}$, $x \in \mathbb{R}^{\infty}_{\geq 0}$
	Output:	TRUE iff $\mathcal{I}(\mathcal{K}) \leq x$
$\textsc{Lower}_{\mathcal{I}}$	**Input**:	$\mathcal{K} \in \mathbb{K}$, $x \in \mathbb{R}^{\infty}_{\geq 0} \setminus \{0\}$
	Output:	TRUE iff $\mathcal{I}(\mathcal{K}) \geq x$

In other words, $\textsc{Exact}_{\mathcal{I}}$ is the problem of deciding whether a given value x is the inconsistency value of a given knowledge base. The problems $\textsc{Upper}_{\mathcal{I}}$ and $\textsc{Lower}_{\mathcal{I}}$ are about deciding whether a given value x is an upper/lower bound of the inconsistency value of a given knowledge base, respectively.

Furthermore, we consider the following natural function problem:

$\textsc{Value}_{\mathcal{I}}$	**Input**:	$\mathcal{K} \in \mathbb{K}$
	Output:	The value of $\mathcal{I}(\mathcal{K})$

Table 5 gives an overview of the computational complexity landscape of the considered measures. Proofs of the results pertaining to $\mathcal{I}_d, \mathcal{I}_{\mathsf{MI}}, \mathcal{I}_{\mathsf{MI^C}}, \mathcal{I}_\eta, \mathcal{I}_c, \mathcal{I}_{mc}, \mathcal{I}_p, \mathcal{I}_{hs}, \mathcal{I}^{\text{hit}}_{\text{dalal}}, \mathcal{I}^{\Sigma}_{\text{dalal}}, \mathcal{I}^{\max}_{\text{dalal}}, \mathcal{I}_{nc}, \mathcal{I}_{mcsc}, \mathcal{I}_{\text{forget}}, \mathcal{I}_{CC}, \mathcal{I}_{\text{is}}$ can be found in [43], see also [28, 45] for proofs pertaining to some generalisations of $\mathcal{I}_c$. Proofs pertaining to the measure $\mathcal{I}_{mv}$ can be found in [46]. Proofs pertaining to the measures $\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$, $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$, and $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$ for the problems $\textsc{Exact}_{\mathcal{I}}$, $\textsc{Upper}_{\mathcal{I}}$, and $\textsc{Lower}_{\mathcal{I}}$ can be found in [40]. Proofs pertaining to the measures $\mathcal{I}_{D_f}$ and $\mathcal{I}_{P_m}$ can be found in the appendix. We refer to [32] and [44, 43] for the exact definitions of the mentioned complexity classes, which will only informally be discussed below.

The analysis of the computational complexity of different measures shows that measuring inconsistency can be significantly more or less complex depending on the actual measure. In general, measures can be categorised into four different classes [43]. The first class contains measures on the first level of the polynomial hierarchy, i. e., those where the problem $\textsc{Upper}_{\mathcal{I}}$ is NP-complete. Under standard complexity-theoretic assumptions (such as assuming that $\mathsf{P} \neq \mathsf{NP}$) these measures are significantly easier to deal with than the other measures. In particular, the decision problem $\textsc{Upper}_{\mathcal{I}}$ itself is not harder than a satisfiability test in propositional logic and implementations for these measures may benefit from the use of SAT solvers. The next class contains measures on the second level of the polynomial hierarchy, i. e., those where the problem $\textsc{Upper}_{\mathcal{I}}$ is Π_2^p-complete. The increase in complexity here is similar (roughly) to the increase in complexity when going from the satisfiability problem in propositional logic to the satisfiability problem in e. g. disjunctive logic programs under the answer set semantics [8]. Solvers for the latter could also be used for the development of implementations for those measures. The measure $\mathcal{I}_{CC}$ is presumably not contained in this second class (although a formal proof is still missing) but Table 5 shows that it is at most on the third level of the polynomial hierarchy, thus again presumably significantly more complex than the previous measures. The final class contains measures beyond the polynomial hierarchy (under standard complexity-theoretic assumptions), i. e., the remaining measures. These measures are inherently more complex as they need to count structures of exponential number (therefore, most of them can be shown to be complete for some "counting" complexity class, those with prefix C). For example, the measure $\mathcal{I}_{\mathsf{MI}}$ is defined to be the number of minimal inconsistent subsets of the knowledge base. This task is hard for two reasons: first, the number of minimal inconsistent subsets may be exponential in the size of the knowledge base, and second, verifying whether some set is indeed a minimal inconsistent subset is hard itself, in fact it is D_1^p-complete [33]. However, for minimal inconsistent sets there are systems available such as [29, 26] that allow the enumeration of those in an effective manner for reasonable problem sizes.

In general, inconsistency measurement is an inherently intractable problem. As the problem of recognising inconsistency is already on the first level of the polynomial hierarchy (it is coNP-complete), we cannot hope for efficient algorithms to *measure* inconsistency (unless $\mathsf{P} = \mathsf{NP}$). But this also means that measures in the first category from above are *optimal* wrt. computational complexity (there are minor complexity-theoretic differences in problems other than $\textsc{Upper}_{\mathcal{I}}$ for these measures, but this is, arguably, negligible compared to the increase in complexity when moving to the second category of measures).

Implementations of the inconsistency measures discussed in this chapter can also be found in the *Tweety Libraries for Artificial Intelligence* [42] and an online interface is available as well[9].

[9] `http://tweetyproject.org/w/incmes`

7 Discussion

Inconsistency measurement is a problem that is not easily defined in a formal manner. Many approaches have been proposed, in particular in recent years, each taking a different perspective on this issue. In this chapter, we addressed the issue of evaluating the appropriateness of these different approaches by considering three different evaluation metrics. First, we discussed rationality postulates. Those aim at prescribing general desirable behaviour of an inconsistency measure and there have also been a lot of proposals in the recent past. Many of them are mutually exclusive, describe orthogonal requirements, and are not generally accepted in the community. Second, we discussed the expressivity of inconsistency measures, i. e., the capability of an inconsistency measure to discriminate between many inconsistent knowledge bases. In general, we expect inconsistency measures to be sensitive towards the addition and deletion of inconsistent parts, so a high expressivity can be regarded as a favourable argument for an inconsistency measure. Finally, we discussed the computational complexity of determining inconsistency values. As deciding inconsistency is (presumably) an intractable problem itself, the task of measuring inconsistency cannot be easier than that. Still, there are differences in the computational complexity of different approaches and it usually better to focus on approaches which are on e. g. the lower levels of the polynomial hierarchy. In order to illustrate the behaviour of these three evaluation metrics, we evaluated a selection of 22 inconsistency measures from the recent literature wrt. those.

This chapter is not intended to identify the best inconsistency measure currently available, but only to highlight their advantages and disadvantages. Different measures behave differently wrt. to the evaluation metrics. The measure $\mathcal{I}_c$ is computationally attractive but its expressivity is limited by the size of the vocabulary. The measure $\mathcal{I}^{\Sigma}_{\mathrm{dalal}}$ has maximal expressivity but fails to satisfy the DO rationality postulate. From these observations, only few general assessments on the quality of each measure can be given. In particular, expressivity and computational complexity are objective evaluation metrics. If given two inconsistency measures $\mathcal{I}_1$ and $\mathcal{I}_2$ with identical behaviour, except that $\mathcal{I}_1$ has strictly higher expressivity or strictly lower computational complexity than $\mathcal{I}_2$, then $\mathcal{I}_1$ should be preferred to $\mathcal{I}_2$ (abstractly speaking). On the other hand, rationality postulates are a subjective means to evaluate inconsistency measures as the appropriateness of many of those is not generally agreed upon, see e. g. [3]. Tables 2 and 3 showed that the behaviour of the evaluated measures differs significantly in light of the available rationality postulates. But in contrast to expressivity and computational complexity, rationality postulates actually address the underlying issue of formally defining “severity of inconsistency”. As a generally agreed upon definition is still an open question, the rationality postulates discussed in this chapter can still serve as a guideline to select an appropriate inconsistency measure wrt. some application. If an application demands the satisfaction of one or more given rationality postulates, among all measures that satisfy those postulates one can select a measure that behaves well wrt. the other evaluation criteria expressivity and computational complexity.

This survey points to a series of open research questions that may be interesting to pursue. For example, the discussion on the "right" set of postulates is not over. The analysis on the compliance of rationality postulates showed that for all postulates we can find an inconsistency measures that satisfies it and another one that violates it. Of course, this situation will only worsen the more measures and postulates are being proposed. What is needed is a characterising definition of an inconsistency measure using few postulates, as the *entropy* is characterised by few simple properties as an information measure. However, we are currently far away from a complete understanding of what an inconsistency constitutes.

Furthermore, our analysis of computational complexity showed that inconsistency measurement may be significantly harder than inconsistency detection (under the usual complexity theoretical assumptions). So far, the *algorithmic study* of inconsistency measurement has (almost) not been investigated at all. Although straightforward prototype implementations of most measures are available (see the remark at the end of the previous section), those implementations do not necessarily optimise runtime performance. Only a few papers [27, 28, 29, 39] have addressed this challenge previously, mainly by developing approximation algorithms. Besides more work on approximation algorithms, another venue for future work is also to develop algorithms that work effectively on certain language fragments—such as certain description logics—and thus may work well in practical applications.

Although we surveyed a rather large selection of inconsistency measures, the analysis is, of course, not complete. Incorporating recent works such as [6, 2, 19] may shed some new light on the issues discussed in this chapter.

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Appendix: Proofs of Technical Results

Examples 4–6 give counterexamples for some false claims given in the literature, see Tables 2 and 3. General corrections regarding the postulates AT and EC can be found in Theorem 6 below.

Example 4. $\mathcal{I}_{P_m}$ does not satisfy CO as falsely claimed in [21]. Consider the knowledge base $\mathcal{K} = \{\neg(a \wedge a), a\}$ where $\{a\}$ is the only minimal proof of a and there is no minimal proof for $\neg a$ (as $\{\neg(a \wedge a), a\}$ does not contain $\neg a$ as a literal). It follows $|P_m^{\mathcal{K}}(a))| = 1$, $|P_m^{\mathcal{K}}(\neg a)| = 0$ and therefore $\mathcal{I}_{P_m}(\mathcal{K}) = 0$, despite the fact that $\mathcal{K}$ is inconsistent.

Example 5. $\mathcal{I}_{nc}$ does not satisfy IN, SI, and DO as falsely claimed in [41]. For SI, consider the knowledge base $\mathcal{K} = \{a \wedge \neg a\}$ and observe that b is a safe formula in $\mathcal{K} \cup \{b\}$. However, we have $\mathcal{I}_{nc}(\mathcal{K}) = 1$ and $\mathcal{I}_{nc}(\mathcal{K} \cup \{b\}) = 2$ contradicting SI. Due to Theorem 1 $\mathcal{I}_{nc}$ cannot satisfy IN as well. For DO, consider the knowledge base $\mathcal{K}' = \{a, a \wedge a, a \wedge a \wedge a, \neg a\}$, the formula $\alpha = \neg a$, and the formula $\beta = \neg a \wedge \neg a$. Observe that $\alpha \not\models \perp$ and $\alpha \models \beta$. We have

$$\mathcal{I}_{nc}(\mathcal{K}' \cup \{\alpha\}) = \mathcal{I}_{nc}(\{a, a \wedge a, a \wedge a \wedge a, \neg a\}) = 3$$
$$\mathcal{I}_{nc}(\mathcal{K}' \cup \{\beta\}) = \mathcal{I}_{nc}(\{a, a \wedge a, a \wedge a \wedge a, \neg a, \neg a \wedge \neg a\}) = 4$$

contradicting DO for $\mathcal{I}_{nc}$.

Example 6. $\mathcal{I}^{\text{hit}}_{\text{dalal}}$ and $\mathcal{I}^{\Sigma}_{\text{dalal}}$ do not satisfy DO as falsely claimed in [11, 12]. Consider the knowledge base $\mathcal{K} = \{a, a \wedge a, a \wedge a \wedge a, \neg a\}$, the formula $\alpha = \neg a$, and the formula $\beta = \neg a \wedge \neg a$. Observe that $\alpha \not\models \perp$ and $\alpha \models \beta$. We have

$$\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K} \cup \{\alpha\}) = \mathcal{I}^{\text{hit}}_{\text{dalal}}(\{a, a \wedge a, a \wedge a \wedge a, \neg a\}) = 1$$
$$\mathcal{I}^{\text{hit}}_{\text{dalal}}(\mathcal{K} \cup \{\beta\}) = \mathcal{I}^{\text{hit}}_{\text{dalal}}(\{a, a \wedge a, a \wedge a \wedge a, \neg a, \neg a \wedge \neg a\}) = 2$$

$$\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K} \cup \{\alpha\}) = \mathcal{I}^{\Sigma}_{\text{dalal}}(\{a, a \wedge a, a \wedge a \wedge a, \neg a\}) = 1$$
$$\mathcal{I}^{\Sigma}_{\text{dalal}}(\mathcal{K} \cup \{\beta\}) = \mathcal{I}^{\Sigma}_{\text{dalal}}(\{a, a \wedge a, a \wedge a \wedge a, \neg a, \neg a \wedge \neg a\}) = 2$$

contradicting DO for both $\mathcal{I}^{\text{hit}}_{\text{dalal}}$ and $\mathcal{I}^{\Sigma}_{\text{dalal}}$.

We now provide proofs for the missing statements regarding the compliance of the rationality postulates of the measures $\mathcal{I}_{mcsc}$, $\mathcal{I}_{\text{forget}}$, $\mathcal{I}_{CC}$, and $\mathcal{I}_{\text{is}}$; see the Tables 2 and 3. For all proofs in the Appendix we denote by +X a proof that shows that property X is satisfied and by −X a proof that shows that property X is violated.

Theorem 2. $\mathcal{I}_{mcsc}$ satisfies SI, EC, FD, and SY. $\mathcal{I}_{mcsc}$ does not satisfy NO, PY, MN, AT, AC, CD, EX, and AI.

Proof.

−**NO** We have $\mathcal{I}_{mcsc}(\{a, \neg a\}) = 2$, so $\mathcal{I}_{mcsc}$ violates NO.

+**SI** This follows from IN due to Theorem 1.

−**PY** Consider $\mathcal{K} = \{a, b, \neg a, \neg b, a \vee b\}$ and observe that $a \vee b \notin \mathsf{Free}(\mathcal{K})$. However, we have $\mathcal{I}_{mcsc}(\mathcal{K}) = \mathcal{I}_{mcsc}(\mathcal{K} \setminus \{a \vee b\}) = 4$.

−**MN** Proposition 2 in [1] showed that $\mathcal{I}_{mcsc}(M) = 2$ for minimal inconsistent sets M with $|M| > 1$.

−**AT** Consider the minimal inconsistent sets $M = \{a, \neg a\}$ and $M' = \{a, b, \neg a \vee \neg b\}$. We have $|M| < |M'|$ but $\mathcal{I}_{mcsc}(M) = \mathcal{I}_{mcsc}(M') = 2$.

+**EC** For any minimal inconsistent set M with $|M| > 1$ we have $\mathcal{I}_{mcsc}(M) = 2$ due to Proposition 6 in [1]. If $|M| = 1$ we have $\mathcal{I}_{mcsc}(M) = 1$.

−**AC** Consider $M_i = \{a_1, \ldots, a_i, \neg a_1 \vee \ldots \vee \neg a_i\}$ for $i \in \mathbb{N}$. Then $\lim_{i\to\infty} |M_i| = \infty$ but $\lim_{i\to\infty} \mathcal{I}_{mcsc}(M_i) = 2$.

−**CD** We have $\mathcal{I}_{mcsc}(\{a \wedge \neg a, b \wedge \neg b\}) = 2$ but every non-empty subset of $\{a \wedge \neg a, b \wedge \neg b\}$ is inconsistent.

+**FD** This follows from MO due to Theorem 1.

+**SY** Let $\mathcal{K}, \mathcal{K}'$ be knowledge bases with $\mathcal{K} \equiv_b \mathcal{K}'$ and let s be a bijection $s : \mathcal{K} \to \mathcal{K}'$ such that $\alpha \equiv s(\alpha)$ for all $\alpha \in \mathcal{K}$. Then $|\mathcal{K}| = |\mathcal{K}'|$ and observe that $\{\alpha_1, \ldots, \alpha_k\} \subseteq \mathcal{K}$ is a consistent set if and only if $\{s(\alpha_1), \ldots, s(\alpha_k)\} \subseteq \mathcal{K}'$ is a consistent set. It follows that $\mathcal{I}_{mcsc}(\mathcal{K}) = \mathcal{I}_{mcsc}(\mathcal{K}')$.

−**EX** Consider $\mathcal{K} = \{a \wedge \neg a\}$, $\mathcal{K}' = \{b \wedge c\}$, and $\mathcal{K}'' = \{b, c\}$ and observe that $\mathcal{K}' \equiv \mathcal{K}''$. However, we have $\mathcal{I}_{mcsc}(\mathcal{K} \cup \mathcal{K}') = 2 \neq 3 = \mathcal{I}_{mcsc}(\mathcal{K} \cup \mathcal{K}'')$.

−**AI** We have $\mathcal{I}_{mcsc}(\{a \wedge \neg a\}) = 1 \neq 2 = \mathcal{I}_{mcsc}(\{a, \neg a\})$.

□

Theorem 3. $\mathcal{I}_{\text{forget}}$ satisfies SI, SA, and FD. $\mathcal{I}_{\text{forget}}$ does not satisfy NO, PY, MI, MN, AT, EC, AC, CD, and SY.

Proof.

−**NO** We have $\mathcal{I}_{\text{forget}}(\{a \wedge \neg a, b \wedge \neg b\}) = 2$, so $\mathcal{I}_{\text{forget}}$ violates NO.

+**SI** This follows from IN due to Theorem 1.

+**SA** Let $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ with $\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset$ and $\mathcal{I}_{\text{forget}}(\mathcal{K}) = k$. Let $a_1, \ldots, a_k \in \mathsf{At}$, $i_1, \ldots, i_k \in \mathbb{N}$, and $\phi_1, \ldots, \phi_k \in \{\bot, \top\}$ be such that

$$(\bigwedge \mathcal{K})[a_1, i_1 \to \phi_1; \ldots; a_k, i_k \to \phi_k] \not\models \bot$$

As each triple (a_j, i_j, ϕ_j) for $j = 1, \ldots, k$ identifies a replacement in either $\mathcal{K}_1$ or $\mathcal{K}_2$ we can write the above as

$$(\bigwedge \mathcal{K}_1)[a_1, i_1 \to \phi_1; \ldots; a_{k'}, i_{k'} \to \phi_{k'}] \wedge$$
$$(\bigwedge \mathcal{K}_2)[a_{k'+1}, i_{k'+1} \to \phi_{k'+1}; \ldots; a_k, i_k \to \phi_k] \not\models \bot$$

assuming that the $a_1, \ldots, a_k$ are numbered adequately and $1 \leq k' \leq k$. It follows that

$$(\bigwedge \mathcal{K}_1)[a_1, i_1 \to \phi_1; \ldots; a_{k'}, i_{k'} \to \phi_{k'}] \not\models \perp \qquad \text{and}$$
$$(\bigwedge \mathcal{K}_2)[a_{k'+1}, i_{k'+1} \to \phi_{k'+1}; \ldots; a_k, i_k \to \phi_k] \not\models \perp$$

and therefore $\mathcal{I}_{\text{forget}}(\mathcal{K}_1) \leq k'$ and $\mathcal{I}_{\text{forget}}(\mathcal{K}_2) \leq k - k'$ and therefore $\mathcal{I}_{\text{forget}}(\mathcal{K}) \geq \mathcal{I}_{\text{forget}}(\mathcal{K}_1) + \mathcal{I}_{\text{forget}}(\mathcal{K}_2)$.

- −**PY** We have $\mathcal{I}_{\text{forget}}(\{a, \neg a\}) = \mathcal{I}_{\text{forget}}(\{a, a \wedge a, \neg a\}) = 1$ but $a \wedge a \notin \mathsf{Free}(\{a, a \wedge a, \neg a\})$.
- −**MI** Consider $\mathcal{K} = \{a \wedge \neg a \wedge c\}$ and $\mathcal{K}' = \{b \wedge \neg b \wedge \neg c\}$ and observe that $\mathsf{MI}(\mathcal{K} \cup \mathcal{K}') = \{\mathcal{K}, \mathcal{K}'\}$, $\mathsf{MI}(\mathcal{K}) = \{\mathcal{K}\}$, $\mathsf{MI}(\mathcal{K}') = \{\mathcal{K}'\}$ but $\mathcal{I}_{\text{forget}}(\mathcal{K} \cup \mathcal{K}') = 3 \neq 2 = 1 + 1 = \mathcal{I}_{\text{forget}}(\mathcal{K}) + \mathcal{I}_c(\mathcal{K}')$.
- −**MN** We have $\mathcal{I}_{\text{forget}}(\{a \wedge b, \neg a \wedge \neg b\}) = 2$ but $\{a \wedge b, \neg a \wedge \neg b\}$ is minimally inconsistent.
- −**AT** Consider the minimal inconsistent sets $M = \{a \wedge \neg a\}$ and $M' = \{a, \neg a\}$. We have $|M| < |M'|$ but $\mathcal{I}_{\text{forget}}(M) = \mathcal{I}_{\text{forget}}(M') = 1$.
- −**EC** Consider the minimal inconsistent sets $M = \{a \wedge \neg a\}$ and $M' = \{a \wedge b \wedge \neg a \wedge \neg b\}$. We have $|M| = |M'|$ but $\mathcal{I}_{\text{forget}}(M) = 1 < 2 = \mathcal{I}_{\text{forget}}(M')$.
- −**AC** Consider $M_i = \{a_1, \ldots, a_i, \neg a_1 \vee \ldots \vee \neg a_i\}$ for $i \in \mathbb{N}$. Then $\lim_{i \to \infty} |M_i| = \infty$ but $\lim_{i \to \infty} \mathcal{I}_{\text{forget}}(M_i) = 1$.
- −**CD** We have $\mathcal{I}_{\text{forget}}(\{a, \neg a\}) = 1$ but $\{a\} \subseteq \{a, \neg a\}$ is consistent.
- +**FD** This follows from MO due to Theorem 1.
- −**SY** Consider $\mathcal{K} = \{a \wedge \neg a\}$ and $\mathcal{K}' = \{a \wedge \neg a \wedge b \wedge \neg b\}$. Then $\mathcal{K} \equiv_b \mathcal{K}'$ but $\mathcal{I}_{\text{forget}}(\mathcal{K}) = 1 \neq 2 = \mathcal{I}_{\text{forget}}(\mathcal{K}')$.

□

Theorem 4. $\mathcal{I}_{CC}$ satisfies CO, IN, SI, MN, EC, FD, and SY. $\mathcal{I}_{CC}$ does not satisfy NO, DO, PY, AT, AC, CD, EX, and AI.

Proof.

- +**CO** For consistent $\mathcal{K}$ the set $\emptyset$ is the only conditional independent MUS partition of $\mathcal{K}$ and therefore $\mathcal{I}_{CC}(\mathcal{K}) = 0$. For inconsistent $\mathcal{K}$, any set $\{M\}$ with $M \in \mathsf{MI}(\mathcal{K})$ is a conditional independent MU partition of $\mathcal{K}$ and therefore $\mathcal{I}_{CC}(\mathcal{K}) \geq 1$.
- −**NO** We have $\mathcal{I}_{CC}(\{a, \neg a, b, \neg b\}) = 2$, so $\mathcal{I}_{CC}$ violates NO.

+**IN** This follows from the fact that S is a conditional independent MUS partition of $\mathcal{K}$ if S is a conditional independent MUS partition of $\mathcal{K} \setminus \{\alpha\}$ for $\alpha \in \mathsf{Free}(\mathcal{K} \setminus \{\alpha\})$.

−**DO** Consider $\mathcal{K} = \{a, \neg a \wedge b, \neg c\}$, $\alpha = \neg b \wedge c$ and $\beta = c$. Observe $\alpha \not\models \perp$ and $\alpha \models \beta$. However, we have $\mathcal{I}_{CC}(\mathcal{K} \cup \{\alpha\}) = 1 < 2 = \mathcal{I}_{CC}(\mathcal{K} \cup \{\beta\})$.

+**SI** This follows from IN due to Theorem 1.

−**PY** We have $\mathcal{I}_{CC}(\{a, \neg a, a \wedge a\}) = \mathcal{I}_{CC}(\{a, \neg a\}) = 1$ but $a \wedge a \notin \mathsf{Free}(\{a, \neg a, a \wedge a\})$.

+**MN** For a minimal inconsistent M the set $\{M\}$ is the maximal conditional independent MUS partition of M and therefore $\mathcal{I}_{CC}(M) = 1$.

−**AT** This follows from MN due to Theorem 1.

+**EC** This follows from MN due to Theorem 1.

−**AC** This follows from MN due to Theorem 1.

−**CD** This follows from MN due to Theorem 1.

+**FD** This follows from MO due to Theorem 1.

+**SY** Let $\mathcal{K}, \mathcal{K}'$ be knowledge bases with $\mathcal{K} \equiv_b \mathcal{K}'$ and let s be a bijection $s : \mathcal{K} \to \mathcal{K}'$ such that $\alpha \equiv s(\alpha)$ for all $\alpha \in \mathcal{K}$. Then $|\mathcal{K}| = |\mathcal{K}'|$ and observe that $\{M_1, \ldots, M_k\}$ is a conditionally independent MUS partition of $\mathcal{K}$ if and only if

$$\{\{s(\alpha) \mid \alpha \in M_i\} \mid i = 1, \ldots, k\}$$

is a conditionally independent MUS partition of $\mathcal{K}'$. It follows that $\mathcal{I}_{mcsc}(\mathcal{K}) = \mathcal{I}_{mcsc}(\mathcal{K}')$.

−**EX** Consider $\mathcal{K} = \{\neg a, \neg b\}$, $\mathcal{K}' = \{a, b\}$, and $\mathcal{K}'' = \{a \wedge b\}$. Observe $\mathcal{K}' \equiv \mathcal{K}''$ but $\mathcal{I}_{CC}(\mathcal{K} \cup \mathcal{K}') = 2 \neq 1 = \mathcal{I}_{CC}(\mathcal{K} \cup \mathcal{K}'')$.

−**AI** The counterexample for EX above also serves as a counterexample for AI.

□

Theorem 5. $\mathcal{I}_{\text{is}}$ satisfies SI, SA, PY, EC, FD, and SY. $\mathcal{I}_{\text{is}}$ does not satisfy NO, DO, AT, AC, CD, EX, and AI.

Proof. −**NO** We have $\mathcal{I}_{\text{is}}(\{a, \neg a, b, \neg b\}) = \ln 4 \approx 1.39$, so $\mathcal{I}_{\text{is}}$ violates NO.

−**DO** Consider $\mathcal{K} = \{a, \neg a\}$, $\alpha = a$ and $\beta = a \wedge a$. Observe $\alpha \not\models \perp$ and $\alpha \models \beta$. However, we have $\mathcal{I}_{\text{is}}(\mathcal{K} \cup \{\alpha\}) = 1 < \ln 3 = \mathcal{I}_{\text{is}}(\mathcal{K} \cup \{\beta\})$.

+**SI** This follows from IN due to Theorem 1.

+**SA** Let $\mathcal{K}, \mathcal{K}'$ with $\mathcal{K} \cap \mathcal{K}' = \emptyset$. Then $\mathsf{MI}(\mathcal{K}) \cap \mathsf{MI}(\mathcal{K}') = \emptyset$ and $\mathsf{MI}(\mathcal{K}) \cup \mathsf{MI}(\mathcal{K}') \subseteq \mathsf{MI}(\mathcal{K} \cup \mathcal{K}')$. Then by taking the union of any set of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K})$ and any set of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K}')$ one obtains a set of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K} \cup \mathcal{K}')$ (note that the empty set is a set of pairwise disjoint subsets of both $\mathsf{MI}(\mathcal{K})$ and $\mathsf{MI}(\mathcal{K}')$). If i_S is the number of sets of pairwise disjoint subsets of a set S then $i_{\mathsf{MI}(\mathcal{K} \cup \mathcal{K}')} \geq i_{\mathsf{MI}(\mathcal{K})} i_{\mathsf{MI}(\mathcal{K}')}$. Therefore

$$\begin{aligned}\mathcal{I}_{\text{is}}(\mathcal{K} \cup \mathcal{K}') &= \ln i_{\mathsf{MI}(\mathcal{K} \cup \mathcal{K}')} \\ &\geq \ln i_{\mathsf{MI}(\mathcal{K})} i_{\mathsf{MI}(\mathcal{K}')} \\ &= \ln i_{\mathsf{MI}(\mathcal{K})} + \ln i_{\mathsf{MI}(\mathcal{K}')} \\ &= \mathcal{I}_{\text{is}}(\mathcal{K}) + \mathcal{I}_{\text{is}}(\mathcal{K}')\end{aligned}$$

+**PY** If $\alpha \notin \mathsf{Free}(\mathcal{K})$ then $\mathsf{MI}(\mathcal{K} \setminus \{\alpha\}) \subsetneq \mathsf{MI}(\mathcal{K})$. Then every set of pairwise disjoint subsets S of $\mathsf{MI}(\mathcal{K} \setminus \{\alpha\})$ is also a set of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K})$. Let $M \in \mathsf{MI}(\mathcal{K})$ with $\alpha \in M$. Then $M \notin \mathsf{MI}(\mathcal{K} \setminus \{\alpha\})$ and $\{M\}$ is a set of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K})$. Therefore, the set of sets of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K})$ is a strict superset of the set of sets of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K} \setminus \{\alpha\})$. The claim follows from the monotonicity of the logarithm.

−**AT** This follows from MN due to Theorem 1.

+**EC** This follows from MN due to Theorem 1.

−**AC** This follows from MN due to Theorem 1.

−**CD** This follows from MN due to Theorem 1.

+**FD** This follows from MO due to Theorem 1.

+**SY** Let $\mathcal{K}, \mathcal{K}'$ be knowledge bases with $\mathcal{K} \equiv_b \mathcal{K}'$ and let s be a bijection $s : \mathcal{K} \to \mathcal{K}'$ such that $\alpha \equiv s(\alpha)$ for all $\alpha \in \mathcal{K}$. Observe that $M = \{\alpha_1, \ldots, \alpha_k\} \in \mathsf{MI}(\mathcal{K})$ iff $s(M) = \{s(\alpha_1,) \ldots, s(\alpha_k)\} \in \mathsf{MI}(\mathcal{K}')$. It follows that $\{M_1, \ldots, M_l\}$ is a set of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K})$ iff $\{s(M_1), \ldots, s(M_l)\}$ is a set of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K}')$ and therefore the claim.

−**EX** Consider $\mathcal{K} = \{\neg a, \neg b\}$, $\mathcal{K}' = \{a, b\}$, and $\mathcal{K}'' = \{a \wedge b\}$. Observe $\mathcal{K}' \equiv \mathcal{K}''$ but $\mathcal{I}_{\text{is}}(\mathcal{K} \cup \mathcal{K}') = \ln 4 \neq \ln 3 = \mathcal{I}_{\text{is}}(\mathcal{K} \cup \mathcal{K}'')$.

−**AI** The counterexample for EX above also serves as a counterexample for AI.

$\square$

The following theorem corrects some previous statements from [41, 40] where the postulates AT and EC have been stated in different way compared to the original definition from [30]. More precisely, the following results shows the compliance of all considered measures (except $\mathcal{I}_{mcsc}$, $\mathcal{I}_{\text{forget}}$, $\mathcal{I}_{CC}$, and $\mathcal{I}_{\text{is}}$ which have been dealt with above) with the postulates AT and EC. It corrects previous results by showing that 1.) $\mathcal{I}_d$, $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{hs}$, $\mathcal{I}_{mv}$, $\mathcal{I}^{\text{hit}}_{\text{dalal}}$, and $\mathcal{I}_{nc}$ do not satisfy AT, 2.) $\mathcal{I}_{\mathsf{MI}}$ does not satisfy EC, and 3.) $\mathcal{I}_d$, $\mathcal{I}_{mc}$, $\mathcal{I}_{hs}$, $\mathcal{I}^{\text{hit}}_{\text{dalal}}$, $\mathcal{I}_{mv}$, $\mathcal{I}_{nc}$, and $\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$ satisfy EC. All other statements remain unchanged.

Theorem 6. For $\mathcal{I} \in \{\mathcal{I}_{\mathsf{MI}^\mathsf{C}}, \mathcal{I}_\eta, \mathcal{I}_{D_f}\}$, $\mathcal{I}$ satisfies AT. For $\mathcal{I} \in \{\mathcal{I}_d, \mathcal{I}_{\mathsf{MI}}, \mathcal{I}_c, \mathcal{I}_{mc}, \mathcal{I}_p, \mathcal{I}_{hs}, \mathcal{I}^{\Sigma}_{\text{dalal}}, \mathcal{I}^{\max}_{\text{dalal}}, \mathcal{I}^{\text{hit}}_{\text{dalal}}, \mathcal{I}_{P_m}, \mathcal{I}_{mv}, \mathcal{I}_{nc}, \mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}, \mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}, \mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}\}$, $\mathcal{I}$ violates AT. For $\mathcal{I} \in \{\mathcal{I}_d, \mathcal{I}_{\mathsf{MI}}, \mathcal{I}_{\mathsf{MI}^\mathsf{C}}, \mathcal{I}_\eta, \mathcal{I}_{mc}, \mathcal{I}_p, \mathcal{I}_{hs}, \mathcal{I}^{\text{hit}}_{\text{dalal}}, \mathcal{I}_{D_f}, \mathcal{I}_{mv}, \mathcal{I}_{nc}, \mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}\}$, $\mathcal{I}$ satisfies EC. For $\mathcal{I} \in \{\mathcal{I}_c, \mathcal{I}^{\Sigma}_{\text{dalal}}, \mathcal{I}^{\max}_{\text{dalal}}, \mathcal{I}_{P_m}, \mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}, \mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}\}$, $\mathcal{I}$ violates EC.

Proof.

$\mathcal{I}_d$ **−AT** Consider $M = \{a, b, \neg a \vee \neg b\}$ and $M' = \{\neg a, a\}$. We have $|M| > |M'|$ but $\mathcal{I}_d(M) = 1 = \mathcal{I}_d(M')$.

$\mathcal{I}_d$ **+EC** For any pair of minimal inconsistent sets M, M' (independently of whether they have the same cardinality) we always have $\mathcal{I}_d(M) = 1 = \mathcal{I}_d(M')$.

$\mathcal{I}_{\mathsf{MI}}$ **−AT** Consider $M = \{a, b, \neg a \vee \neg b\}$ and $M' = \{\neg a, a\}$. We have $|M| > |M'|$ but $\mathcal{I}_{\mathsf{MI}}(M) = 1 = \mathcal{I}_{\mathsf{MI}}(M')$.

$\mathcal{I}_{\mathsf{MI}}$ **+EC** For any pair of minimal inconsistent sets M, M' (independently of whether they have the same cardinality) we always have $\mathcal{I}_{\mathsf{MI}}(M) = 1 = \mathcal{I}_{\mathsf{MI}}(M')$.

$\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ **+AT** Let M, M' be minimally inconsistent with $|M'| < |M|$. It follows directly that $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}(M) = 1/|M| < 1/|M'| = \mathcal{I}_{\mathsf{MI}^\mathsf{C}}(M')$.

$\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ **+EC** Let M, M' be minimally inconsistent with $|M'| = |M|$. It follows directly that $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}(M) = 1/|M| = 1/|M'| = \mathcal{I}_{\mathsf{MI}^\mathsf{C}}(M')$.

$\mathcal{I}_\eta$ **+AT** In [23] it has been shown (Theorem 2.12, slightly rephrased here) that for any minimal inconsistent M, $\mathcal{I}_\eta(M) = 1/|M|$. Then the proof of AT is analogous to the corresponding proof for $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ (see above).

$\mathcal{I}_\eta$ **+EC** In [23] it has been shown (Theorem 2.12, slightly rephrased here) that for any minimal inconsistent M, $\mathcal{I}_\eta(M) = 1/|M|$. Then the proof of EC is analogous to the corresponding proof for $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ (see above).

$\mathcal{I}_c$ **−AT** Consider $M = \{a, \neg a\}$ and $M' = \{\neg a \wedge a\}$. We have $|M| > |M'|$ but $\mathcal{I}_c(M) = 1 = \mathcal{I}_c(M')$.

$\mathcal{I}_c$ $-$**EC** Consider $M = \{a, \neg a\}$ and $M' = \{a \wedge b, \neg a \wedge \neg b\}$. We have $|M| = |M'|$ but $\mathcal{I}_c(M) = 1 \neq 2 = \mathcal{I}_c(M')$.

$\mathcal{I}_{mc}$ $-$**AT** Consider $M = \{a, \neg a\}$ and $M' = \{a \wedge \neg a\}$. We have $|M| > |M'|$ but $\mathcal{I}_{mc}(M) = 1 = \mathcal{I}_{mc}(M')$.

$\mathcal{I}_{mc}$ $+$**EC** Let $M \in \mathsf{MI}(\mathcal{K})$, if $|M| = 1$ then $\mathcal{I}_{mc}(M) = 1$ and if $|M| > 1$ then $\mathcal{I}_{mc}(M) = |M| - 1$.

$\mathcal{I}_p$ $-$**AT** Consider $M = \{a, b, \neg a \vee \neg b\}$ and $M' = \{\neg a, a\}$. We have $|M| > |M'|$ but $\mathcal{I}_p(M) = 3 > 2 = \mathcal{I}_p(M')$.

$\mathcal{I}_p$ $+$**EC** For any minimally inconsistent M, $\mathcal{I}_p(M) = |M|$.

$\mathcal{I}_{hs}$ $-$**AT** Consider $M = \{a, b, \neg a \vee \neg b\}$ and $M' = \{\neg a, a\}$. We have $|M| > |M'|$ but $\mathcal{I}_{hs}(M) = 1 = \mathcal{I}_{hs}(M')$.

$\mathcal{I}_{hs}$ $+$**EC** For $M \in \mathsf{MI}(\mathcal{K})$ observe that for $|M| = 1$, $\mathcal{I}_{hs}(M) = \infty$ and for $|M| > 1$ we have $\mathcal{I}_{hs}(M) = 1$.

$\mathcal{I}^{\Sigma}_{\mathbf{dalal}}$ $-$**AT** Consider $M = \{a, b, \neg a \vee \neg b\}$ and $M' = \{\neg a, a\}$. We have $|M| > |M'|$ but $\mathcal{I}^{\Sigma}_{\text{dalal}}(M) = 1 = \mathcal{I}^{\Sigma}_{\text{dalal}}(M')$.

$\mathcal{I}^{\Sigma}_{\mathbf{dalal}}$ $-$**EC** Consider $M = \{a, \neg a\}$ and $M' = \{a \wedge b, \neg a \wedge \neg b\}$. We have $|M| = |M'|$ but $\mathcal{I}^{\Sigma}_{\text{dalal}}(M) = 1 \neq 2 = \mathcal{I}^{\Sigma}_{\text{dalal}}(M')$.

$\mathcal{I}^{\max}_{\mathbf{dalal}}$ $-$**AT** Consider $M = \{a, b, \neg a \vee \neg b\}$ and $M' = \{\neg a, a\}$. We have $|M| > |M'|$ but $\mathcal{I}^{\max}_{\text{dalal}}(M) = 1 = \mathcal{I}^{\max}_{\text{dalal}}(M')$.

$\mathcal{I}^{\max}_{\mathbf{dalal}}$ $-$**EC** Consider $M = \{a, \neg a\}$ and $M' = \{a \wedge b \wedge c, \neg a \wedge \neg b \wedge \neg c\}$. We have $|M| = |M'|$ but $\mathcal{I}^{\max}_{\text{dalal}}(M) = 1 \neq 2 = \mathcal{I}^{\max}_{\text{dalal}}(M')$.

$\mathcal{I}^{\mathbf{hit}}_{\mathbf{dalal}}$ $-$**AT** Consider $M = \{a, b, \neg a \vee \neg b\}$ and $M' = \{\neg a, a\}$. We have $|M| > |M'|$ but $\mathcal{I}^{\text{hit}}_{\text{dalal}}(M) = 1 = \mathcal{I}^{\text{hit}}_{\text{dalal}}(M')$.

$\mathcal{I}^{\mathbf{hit}}_{\mathbf{dalal}}$ $+$**EC** Note that $\mathcal{I}^{\text{hit}}_{\text{dalal}}(M) = 1$ for every minimal inconsistent M.

$\mathcal{I}_{D_f}$ $+$**AT** For $M \in \mathsf{MI}(\mathcal{K})$ observe that $\mathcal{I}_{D_f}(M) = 1/|M|$. Then the proof of AT is analogous to the corresponding proof for $\mathcal{I}_{\mathsf{MI^C}}$ (see above).

$\mathcal{I}_{D_f}$ $+$**EC** For $M \in \mathsf{MI}(\mathcal{K})$ observe that $\mathcal{I}_{D_f}(M) = 1/|M|$. Then the proof of EC is analogous to the corresponding proof for $\mathcal{I}_{\mathsf{MI^C}}$ (see above).

$\mathcal{I}_{P_m}$ $-$**AT** Consider $M = \{\neg a, a\}$ and $M' = \{\neg a \wedge a\}$. We have $|M| > |M'|$ but $\mathcal{I}_{P_m}(M) = 1 = \mathcal{I}_{P_m}(M')$.

$\mathcal{I}_{P_m}$ $-$**EC** Consider $M = \{a, \neg a\}$ and $M' = \{a \wedge b, \neg a \wedge \neg b\}$. We have $|M| = |M'|$ but $\mathcal{I}_{P_m}(M) = 1 \neq 2 = \mathcal{I}_{P_m}(M')$.

$\mathcal{I}_{mv}$ $-$**AT** Consider $M = \{\neg a, a\}$ and $M' = \{\neg a \wedge a\}$. We have $|M| > |M'|$ but $\mathcal{I}_{mv}(M) = 1 = \mathcal{I}_{mv}(M')$.

$\mathcal{I}_{mv}$ $+$**EC** Observe $\mathcal{I}_{mv}(M) = 1$ for every minimal inconsistent set M.

$\mathcal{I}_{nc}$ $-$**AT** Consider $M = \{\neg a, a\}$ and $M' = \{\neg a \wedge a\}$. We have $|M| > |M'|$ but $\mathcal{I}_{nc}(M) = 1 = \mathcal{I}_{nc}(M')$.

$\mathcal{I}_{nc}$ $+$**EC** Observe $\mathcal{I}_{nc}(M) = 1$ for every minimal inconsistent set M.

$\mathcal{I}^{\mathbf{fuz}}_{t_{\mathbf{prod}}}$ $-$**AT** Consider $M = \{\neg a, a\}$ and $M' = \{\neg a \wedge a\}$. We have $|M| > |M'|$ but $\mathcal{I}^{\mathrm{fuz}}_{t_{\mathrm{prod}}}(M) = 0.75 = \mathcal{I}^{\mathrm{fuz}}_{t_{\mathrm{prod}}}(M')$.

$\mathcal{I}^{\mathbf{fuz}}_{t_{\mathbf{prod}}}$ $-$**EC** Consider $M = \{a, \neg a\}$ and $M' = \{a \wedge a, \neg a \wedge \neg a\}$. We have $|M| = |M'|$ but $\mathcal{I}^{\mathrm{fuz}}_{t_{\mathrm{prod}}}(M) = 0.75 \neq 0.9375 = \mathcal{I}^{\mathrm{fuz}}_{t_{\mathrm{prod}}}(M')$.

$\mathcal{I}^{\mathbf{fuz},\Sigma}_{t_{\min}}$ $-$**AT** Consider $M = \{\neg a, a\}$ and $M' = \{\neg a \wedge a\}$. We have $|M| > |M'|$ but $\mathcal{I}^{\mathrm{fuz},\Sigma}_{t_{\min}}(M) = 1 = \mathcal{I}^{\mathrm{fuz},\Sigma}_{t_{\min}}(M')$.

$\mathcal{I}^{\mathbf{fuz},\Sigma}_{t_{\min}}$ $+$**EC** Observe that for a minimal inconsistent M with $|M| = 1$, $\mathcal{I}^{\mathrm{fuz},\Sigma}_{t_{\min}}(M) = 1/2$ due to Proposition 3 in [40]. Furthermore, for a minimal inconsistent M with $|M| > 1$ one can see that $\mathcal{I}^{\mathrm{fuz},\Sigma}_{t_{\min}}(M) = 1$ (one can always define a fuzzy minimum interpretation ω in such a way that all but one formula of M are satisfied, i. e., $\omega(\alpha) = 1$, and exactly one formula β has $\omega(\beta) = 0$.

$\mathcal{I}^{\mathbf{fuz},\Sigma}_{t_{\mathbf{prod}}}$ $-$**AT** Consider $M = \{\neg a, a\}$ and $M' = \{\neg a \wedge a\}$. We have $|M| > |M'|$ but $\mathcal{I}^{\mathrm{fuz},\Sigma}_{t_{\mathrm{prod}}}(M) = 1 = \mathcal{I}^{\mathrm{fuz},\Sigma}_{t_{\mathrm{prod}}}(M')$.

$\mathcal{I}^{\mathbf{fuz},\Sigma}_{t_{\mathbf{prod}}}$ $-$**EC** Consider $M = \{a, \neg a\}$ and $M' = \{a \wedge b, \neg a \wedge \neg b\}$. We have $|M| = |M'|$ but $\mathcal{I}^{\mathrm{fuz},\Sigma}_{t_{\mathrm{prod}}}(M) = 1 \neq 1.5 = \mathcal{I}^{\mathrm{fuz},\Sigma}_{t_{\mathrm{prod}}}(M')$.

□

We now provide proofs for the missing statements regarding expressivity of the measures $\mathcal{I}_{mcsc}$, $\mathcal{I}_{\mathrm{forget}}$, $\mathcal{I}_{CC}$, and $\mathcal{I}_{\mathrm{is}}$, see Table 4.

Theorem 7. $\mathcal{C}^v(\mathcal{I}_{mcsc}, n) = \mathcal{C}^p(\mathcal{I}_{mcsc}, n) = \infty$, $\mathcal{C}^f(\mathcal{I}_{mcsc}, n) = n + 1$. For $n > 1$, $\mathcal{C}^l(\mathcal{I}_{mcsc}, n) = \infty$.

Proof. Regarding $\mathcal{C}^v(\mathcal{I}_{mcsc}, n) = \mathcal{C}^p(\mathcal{I}_{mcsc}, n) = \infty$, consider the family

$$\mathcal{K}_i = \{\neg a, a, a \wedge a, \ldots, \underbrace{a \wedge \ldots \wedge a}_{i \text{ times}}\}$$

where each $\mathcal{K}_i$ and each formula in $\mathcal{K}_i$ mentions only a single atom, $i \in \mathbb{N}$. Note that each $\mathcal{K}_i$ contains exactly two maximal consistent subsets, namely

$\{\neg a\}$ and $\{a, a \wedge a, \ldots\}$. Those two also comprise the single maximal MC cover (which has an empty intersection). It follows that $\mathcal{I}_{mcsc}(\mathcal{K}_i) = |\mathcal{K}_i| - 0 = i+1$.

Regarding $\mathcal{C}^f(\mathcal{I}_{mcsc}, n) = n+1$ note that $\mathcal{I}_{mcsc}$ is integer-valued and $\mathcal{I}_{mcsc}(\mathcal{K}) \leq |\mathcal{K}|$ by definition, showing that $\mathcal{C}^f(\mathcal{I}_{mcsc}, n) \leq n+1$. To see that $\mathcal{C}^f(\mathcal{I}_{mcsc}, n) \geq n+1$ consider $\mathcal{K}_1, \ldots, \mathcal{K}_{n-1}$ from above showing that $\{2, \ldots, n\}$ are possible values for $\mathcal{I}_{mcsc}$ on knowledge bases of size n or smaller. Furthermore, we have $\mathcal{I}_{mcsc}(\emptyset) = 0$ and $\mathcal{I}_{mcsc}(\{a \wedge \neg a\}) = 1$, yielding $\mathcal{C}^f(\mathcal{I}_{mcsc}, n) \geq n+1$.

Regarding $\mathcal{C}^l(\mathcal{I}_{mcsc}, n) = \infty$ for $n > 1$, consider the family

$$\mathcal{K}'_i = \{a_1, \neg a_1, \ldots, a_i, \neg a_i\}$$

and observe that $\mathcal{I}_{mcsc}(\mathcal{K}'_i) = 2i$.

□

Theorem 8. $\mathcal{C}^v(\mathcal{I}_{\text{forget}}, n) = \mathcal{C}^f(\mathcal{I}_{\text{forget}}, n) = \mathcal{C}^p(\mathcal{I}_{\text{forget}}, n) = \infty$. For $n > 1$, $\mathcal{C}^l(\mathcal{I}_{\text{forget}}, n) = \infty$.

Proof. Regarding $\mathcal{C}^v(\mathcal{I}_{\text{forget}}, n) = \mathcal{C}^f(\mathcal{I}_{\text{forget}}, n) = \mathcal{C}^p(\mathcal{I}_{\text{forget}}, n) = \infty$, consider the family

$$\mathcal{K}_i = \{\underbrace{a \wedge \ldots \wedge a}_{i \text{ times}} \wedge \underbrace{\neg a \wedge \ldots \wedge \neg a}_{i \text{ times}}\}$$

where each $\mathcal{K}_i$ mentions only a single atom and consists of a single formula, $i \in \mathbb{N}$. Observe $\mathcal{I}_{\text{forget}}(\mathcal{K}_i) = i$.

Regarding $\mathcal{C}^l(\mathcal{I}_{\text{forget}}, n) = \infty$ for $n > 1$, consider the family

$$\mathcal{K}'_i = \{a_1, \neg a_1, \ldots, a_i, \neg a_i\}$$

and observe that $\mathcal{I}_{\text{forget}}(\mathcal{K}'_i) = i$. □

Theorem 9. $\mathcal{C}^v(\mathcal{I}_{CC}, n) = \mathcal{C}^p(\mathcal{I}_{CC}, n) = \infty$, $\mathcal{C}^f(\mathcal{I}_{CC}, n) = n+1$. For $n > 1$, $\mathcal{C}^l(\mathcal{I}_{CC}, n) = \infty$.

Proof. Regarding $\mathcal{C}^v(\mathcal{I}_{CC}, n) = \infty$, consider the family

$$\mathcal{K}_i = \{a \wedge \neg a, a \wedge a \wedge \neg a, \ldots, \underbrace{a \wedge \ldots \wedge a}_{i \text{ times}} \wedge \neg a\}$$

with $\mathcal{I}_{CC}(\mathcal{K}_i) = i$.

Regarding $\mathcal{C}^f(\mathcal{I}_{CC}, n) = n+1$, observe that $\mathcal{I}_{CC}$ is integer-valued. Furthermore, $\mathcal{I}_{CC}(\mathcal{K}) \leq |\mathcal{K}|$ as any CI partition $\{K_1, \ldots, K_n\}$ of $\mathcal{K}$ must satisfy $K_i \cap K_j \neq \emptyset$ for all i, j and therefore $n \leq |\mathcal{K}|$. It follows that $\mathcal{C}^f(\mathcal{I}_{CC}, n) \leq n+1$. For $\mathcal{C}^f(\mathcal{I}_{CC}, n) \geq n+1$ consider for $i = 0, \ldots, n$ the family

$$\mathcal{K}'_i = \{a_1 \wedge \neg a_1, \ldots, a_i \wedge \neg a_i\}$$

with $|\mathcal{K}'_i| = \mathcal{I}_{CC}(\mathcal{K}'_i) = i$.

Regarding $\mathcal{C}^p(\mathcal{I}_{CC}, n) = \infty$ and $\mathcal{C}^l(\mathcal{I}_{CC}, n) = \infty$ for $n > 1$, consider the family

$$\mathcal{K}''_i = \{a_1, \neg a_1, \ldots, a_i, \neg a_i\}$$

and observe that $\mathcal{I}_{CC}(\mathcal{K}''_i) = i$. □

Theorem 10. $\mathcal{C}^v(\mathcal{I}_{\text{is}}, n) = \mathcal{C}^p(\mathcal{I}_{\text{is}}, n) = \infty$, $\mathcal{C}^f(\mathcal{I}_{\text{is}}, n) \leq 2^{\binom{n}{\lfloor n/2 \rfloor}} + 1$. For $n > 1$, $\mathcal{C}^l(\mathcal{I}_{\text{is}}, n) = \infty$

Proof. Regarding $\mathcal{C}^v(\mathcal{I}_{\text{is}}, n) = \mathcal{C}^p(\mathcal{I}_{\text{is}}, n) = \infty$, consider the family

$$\mathcal{K}_i = \{\neg a, a, a \wedge a, \ldots, \underbrace{a \wedge \ldots \wedge a}_{i \text{ times}}\}$$

where each $\mathcal{K}_i$ and each formula in $\mathcal{K}_i$ mentions only a single atom, $i \in \mathbb{N}$. Note

$$\mathsf{MI}(\mathcal{K}_i) = \{\{\neg a, a\}, \{\neg a, a \wedge a\}, \ldots, \{\neg a, \underbrace{a \wedge \ldots \wedge a}_{i \text{ times}}\}\}$$

It follows that every singleton subset of $\mathsf{MI}(\mathcal{K}_i)$ and the empty set are the only sets of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K}_i)$. Therefore $\mathcal{I}_{\text{is}}(\mathcal{K}_i) = \ln(i + 1)$.

Regarding $\mathcal{C}^f(\mathcal{I}_{\text{is}}, n) \leq 2^{\binom{n}{\lfloor n/2 \rfloor}} + 1$, recall that $\mathcal{C}^f(\mathcal{I}_{\mathsf{MI}}, n) = \binom{n}{\lfloor n/2 \rfloor} + 1$ [38]. More specifically, the number of minimal inconsistent subsets of a knowledge base with at most n formulas is in $\{0, 1, \ldots, \binom{n}{\lfloor n/2 \rfloor}\}$. If a knowledge base has k minimal inconsistent subsets, i. e. $|\mathsf{MI}(\mathcal{K})| = k$, then there are at most 2^k sets of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K})$ (if all minimal inconsistent subsets are pairwise disjoint). Furthermore, the empty set is always a set of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K})$. Therefore, there are between 1 and 2^k sets of pairwise disjoint subsets of $\mathsf{MI}(\mathcal{K})$ (possibly not all values in-between are attained due to combinatorial reasons, but we are only interested in an upper bound here). Taking the case of a consistent knowledge base into account this shows that $\mathcal{C}^f(\mathcal{I}_{\text{is}}, n) \leq 2^{\binom{n}{\lfloor n/2 \rfloor}} + 1$.

Regarding $\mathcal{C}^l(\mathcal{I}_{\text{is}}, n) = \infty$ for $n > 1$, consider the family

$$\mathcal{K}'_i = \{a_1, \neg a_1, \ldots, a_i, \neg a_i\}$$

and observe that $\mathcal{I}_{\text{is}}(\mathcal{K}'_i) = \ln 2^i = i \ln 2$. □

We now provide proofs for the missing statements regarding the computational complexity of the measures $\mathcal{I}_{D_f}$ and $\mathcal{I}_{P_m}$, see Table 5.

Theorem 11. $\textsc{Exact}_{\mathcal{I}_{D_f}}$, $\textsc{Upper}_{\mathcal{I}_{D_f}}$, $\textsc{Lower}_{\mathcal{I}_{D_f}}$ are in PSPACE and $\textsc{Value}_{\mathcal{I}_{D_f}}$ is in $\mathsf{FPSPACE}$.

Proof. It suffices to show that $\mathcal{I}_{D_f}(\mathcal{K})$ can be computed in polynomial space for all $\mathcal{K}$. Note that the set of all values $|\mathsf{MI}^{(i)}(\mathcal{K})|$ and $|\mathsf{CN}^{(i)}(\mathcal{K})|$ for $i = 1, \ldots, |\mathcal{K}|$ can be stored in polynomial space and that $\mathcal{I}_{D_f}(\mathcal{K})$ can be computed from those values in polynomial space. As we can reuse space, we only need to show that computing each $|\mathsf{MI}^{(i)}(\mathcal{K})|$ and $|\mathsf{CN}^{(i)}(\mathcal{K})|$ for each $i = 1, \ldots, |\mathcal{K}|$ needs at most polynomial space. But this is clear, as we can enumerate each subset S of cardinality i (again reusing space), perform a check whether $S \in \mathsf{MI}^{(i)}(\mathcal{K})$ (or $S \in \mathsf{CN}^{(i)}(\mathcal{K})$) and update some counter. Note that $S \in \mathsf{MI}^{(i)}(\mathcal{K})$ and $S \in \mathsf{CN}^{(i)}(\mathcal{K})$ can be verified by enumerating all interpretations and checking for satisfiability (and additionally for $S \in \mathsf{MI}^{(i)}(\mathcal{K})$ checking each subset with one element less for satisfiability). This can all be done in polynomial space. □

Theorem 12. $\textsc{Exact}_{\mathcal{I}_{P_m}}$, $\textsc{Upper}_{\mathcal{I}_{P_m}}$, $\textsc{Lower}_{\mathcal{I}_{P_m}}$ are in PSPACE and $\textsc{Value}_{\mathcal{I}_{P_m}}$ is in $\mathsf{FPSPACE}$.

Proof. It suffices to show that $\mathcal{I}_{P_m}(\mathcal{K})$ can be computed in polynomial space for all $\mathcal{K}$. We now sketch an algorithm for computing $\mathcal{I}_{P_m}(\mathcal{K})$ running in polynomial space. For each proposition a we keep two counters c_a and $c_{\neg a}$ that keeps track of the number of minimal proofs we encountered for a and $\neg a$, respectively. Note that we only need polynomial space to store these counters. Then by reusing space we enumerate each subset S of $\mathcal{K}$ and check for each proposition a whether S is a minimal proof for a and/or $\neg a$, and update the corresponding counter. Note that checking whether a set S is a minimal proof for some α can be done by enumerating all interpretations (one after the other) and checking for entailment. □

	$\mathcal{C}^v(\mathcal{I}$,n)	$\mathcal{C}^f(\mathcal{I}$,n)	$\mathcal{C}^l(\mathcal{I}$,n)	$\mathcal{C}^p(\mathcal{I}$,n)
$\mathcal{I}_d$	2	2	2^*	2
$\mathcal{I}_{\mathsf{MI}}$	∞	$\binom{n}{\lfloor n/2 \rfloor}+1$	∞^*	∞
$\mathcal{I}_{\mathsf{MI^C}}$	∞	$\leq \Psi(n)^\ddagger$	∞^*	∞
$\mathcal{I}_\eta$	$\Phi(2^n)^\dagger$	$\leq \Phi(\binom{n}{\lfloor n/2 \rfloor})^\dagger$	∞^{**}	∞^*
$\mathcal{I}_c$	$n+1$	∞	∞^*	∞
$\mathcal{I}_{mc}$	∞	$\binom{n}{\lfloor n/2 \rfloor}^{**}$	∞^*	∞
$\mathcal{I}_p$	∞	$n+1$	∞^*	∞
$\mathcal{I}_{hs}$	2^n+1	$n+1$	∞^{**}	∞^*
$\mathcal{I}^{\Sigma}_{\text{dalal}}$	∞	∞^*	∞^*	∞
$\mathcal{I}^{\max}_{\text{dalal}}$	$n+2$	∞^*	$\lfloor (n+7)/3 \rfloor^{**}$	$n+2$
$\mathcal{I}^{\text{hit}}_{\text{dalal}}$	∞	$n+1$	∞^*	∞
$\mathcal{I}_{D_f}$	∞	$\leq \Psi(n)^\ddagger$	∞^*	∞
$\mathcal{I}_{P_m}$	∞	∞	∞^*	∞
$\mathcal{I}_{mv}$	$n+1$	∞^*	∞^*	∞
$\mathcal{I}_{nc}$	∞	$n+1$	∞^*	∞
$\mathcal{I}^{\text{fuz}}_{t_{\text{prod}}}$	∞	∞	∞^*	∞
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\min}}$	∞	$n+1$	∞^*	∞
$\mathcal{I}^{\text{fuz},\Sigma}_{t_{\text{prod}}}$	∞	∞	∞^*	∞
$\mathcal{I}_{mcsc}$	∞	$n+1$	∞^*	∞
$\mathcal{I}_{\text{forget}}$	∞	∞	∞^*	∞
$\mathcal{I}_{CC}$	∞	$n+1$	∞^*	∞
$\mathcal{I}_{\text{is}}$	∞	$\leq 2^{\binom{n}{\lfloor n/2 \rfloor}}+1$	∞^*	∞

Table 4: Characteristics of inconsistency measures ($n \geq 1$);
*only for $n > 1$; **only for $n > 3$
$^\dagger\Phi(x)$ is the number of fractions in the Farey series of order x and can be defined as $\Phi(x) = |\{k/l \mid l = 1, \ldots, x, k = 0, \ldots, l\}|$, see e. g. `http://oeis.org/A005728`
$^\ddagger\Psi(n)$ is the number of profiles of monotone Boolean functions of n variables, see e. g. `http://oeis.org/A220880`

	$\textsc{Exact}_{\mathcal{I}}$	$\textsc{Upper}_{\mathcal{I}}$	$\textsc{Lower}_{\mathcal{I}}$	$\textsc{Value}_{\mathcal{I}}$
$\mathcal{I}_d$	$\mathsf{D}_1^p \cap \mathsf{coD}_1^p$	NP-c	coNP-c	FNP-c
$\mathcal{I}_{\mathsf{MI}}$	$\mathsf{C}_=\mathsf{NP}$-h	CNP-c	CNP-c	$\#\cdot$coNP-c
$\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$	$\mathsf{C}_=\mathsf{NP}$-h	CNP-h	CNP-h	$P^{\#\cdot\mathsf{coNP}}$
$\mathcal{I}_\eta$	D_1^p-c	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[n]}$
$\mathcal{I}_c$	D_1^p-c	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$-c
$\mathcal{I}_{mc}$	$\mathsf{C}_=\mathsf{NP}$-h	CNP-c	CNP-c	$\#\cdot$coNP-c†
$\mathcal{I}_p$	D_2^p-c	Π_2^p-c	Σ_2^p-c	$\mathsf{FP}^{\Sigma_2^p[\log n]}$
$\mathcal{I}_{hs}$	D_1^p-c	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$
$\mathcal{I}_{\mathrm{dalal}}^{\Sigma}$	D_1^p-c	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$-c
$\mathcal{I}_{\mathrm{dalal}}^{\max}$	D_1^p-c	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$
$\mathcal{I}_{\mathrm{dalal}}^{\mathrm{hit}}$	D_1^p-c	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$-c
$\mathcal{I}_{D_f}$	PSPACE	PSPACE	PSPACE	FPSPACE
$\mathcal{I}_{P_m}$	PSPACE	PSPACE	PSPACE	FPSPACE
$\mathcal{I}_{mv}$	D_2^p-c	Π_2^p-c	Σ_2^p-c	$\mathsf{FP}^{\Sigma_2^p[\log n]}$
$\mathcal{I}_{nc}$	D_2^p	Π_2^p-c	Σ_2^p-c	$\mathsf{FP}^{\Sigma_2^p[\log n]}$
$\mathcal{I}_{t_{\mathrm{prod}}}^{\mathrm{fuz}}$	D_1^P	NP-c	coNP-c	?
$\mathcal{I}_{t_{\min}}^{\mathrm{fuz},\Sigma}$	D_1^P	NP-c	coNP-c	?
$\mathcal{I}_{t_{\mathrm{prod}}}^{\mathrm{fuz},\Sigma}$	D_1^P	NP-c	coNP-c	?
$\mathcal{I}_{mcsc}$	D_1^p-c	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$
$\mathcal{I}_{\mathrm{forget}}$	D_1^p-c	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$-c
$\mathcal{I}_{CC}$	D_3^p	Π_3^p	Σ_3^p	$\mathsf{FP}^{\Sigma_3^p[\log n]}$
$\mathcal{I}_{\mathrm{is}}$	$\mathsf{C}_=\mathsf{NP}$-h	CNP-c	CNP-c	$\#\cdot$coNP-c‡

Table 5: Computational complexity of the considered inconsistency measures (all statements are membership statements, an additionally attached "-c" ("-h") also indicates completeness (hardness) for the class); we note that all hardness results for $\#\cdot$coNP are under subtractive reductions; †we show complexity of the (minor) variation that omits subtracting one from the result; ‡we consider here the problem variant that does not apply a logarithm on the result; "?" indicates unknown results